Finite Element Modeling for Stress Analysis

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Preface

This book is intended for beginning courses in finite elements (FE) that are oriented toward *users* of the method. The courses envisioned emphasize the behavior of FE and include computational work in which problems are solved by means of commercial software and the computed results are critically examined. The instructor may often sit with students at the computer to offer advice and to monitor their skill in modeling and assessment of results. The courses would use computational problems as vehicles to teach proper use of FE, rather than use FE as a way to solve certain problems. The book presents a modest amount of theory, discusses the nature of FE solutions, offers modeling advice, suggests computational problems, and emphasizes the need for checking the computed results. Problem areas treated are common in mechanical engineering and related disciplines. Suggested computational problems include topics often treated in a second course in stress analysis, such as spinning disks and elastic foundations. The computational problems usually have simple geometry, so that FE may be emphasized rather than details of data preparation. Some instructors especially those who teach more advanced students, may wish to devise problems of a more "real world" nature, despite their greater complexity.

Several commercial FE programs are available for use on microcomputers and workstations. This book is not tailored to any particular FE program and therefore does not discuss the formalisms of input data preparation. Suitable software will have most of the following features: capability in static stress analysis, structural dynamics, vibration, and heat transfer; a good library of elements; some node and element generation capability; help screens; plotting and animation of displaced shapes; contour plotting of computed stresses without nodal averaging. The software must be easy to use, at the expense of versatility if necessary, so that time will not be wasted in learning procedures peculiar to a certain code but having little to do with insight into the FE method.

Many powerful analytical tools are readily available in the form of computer software. Engineers do not have time to study the theory of all these tools, and undergraduates usually study theory with little enthusiasm. For undergraduate and graduate students alike, it appears that study of only the *theory* of FE confers no ability in the *use* of FE. Theory cannot be ignored, however; an engineer must understand the nature of the analytical method as well as the physical nature of the phenomenon to be studied because computer implementation makes it all too easy to choose inappropriate options or push an analytical method beyond its limits of applicability. Fortunately, the user of FE software need not understand all its details. Mainly, the FE user should grasp the physical problem, understand how FE's behave, know the limitations of the theory on which they are based, and be able and willing to check results for correctness. The checking phase relies more on physical understanding of the problem than on knowledge of FE.

The presentation in this book presumes a knowledge of elementary matrix algebra and the level of physical understanding that a *good* student should have after completing a first course in mechanics of materials. This is adequate preparation for a one-semester course in the practice of FE, during which students will inevitably be exposed to concepts of stress analysis not treated in an elementary mechanics of materials course. The understanding they gain by working with these problems will be primarily physical but will be helpful if theory is to be studied subsequently. In my opinion students in a beginning course learn theory only if forced to do so, and then with little understanding of it. Only later, when the nature of a problem area has become familiar, can theory be understood and its practical value appreciated. These remarks are not intended to imply that the book is unsuitable for students who have advanced knowledge of stress analysis theory. In my experience, a student at any level may be deficient in physical understanding, and graduate students make many of the modeling mistakes also made by undergraduates.

The beginning course I teach is taken by seniors. We currently discuss most of Chapters 1 through 7 and the first four articles of Chapter 9. Isoparametric elements and Sections 5.5 and 6.6 are omitted. For this course I find that previous exposure to the theories of elasticity, plates, shells, and vibrations is not necessary because the essential physical behavior of such problems is easily grasped: flat plates can stretch or bend; curved plates (shells) can simultaneously stretch and bend; examples of vibration are commonplace (e.g., a bell). If courses in these areas were prerequisites, few would enroll in the FE course. Students would then have education in neither FE nor problems to which FE analyis is applied. Yet after graduation they will use FE whether or not they are prepared to do so.

In addition to serving as the primary text in a first FE course, the book should be useful as an adjunct text in a second FE course that considers theory in more detail, and in other courses such as vibrations where the solution of practical problems is considered important. It is in this context that the latter part of Chapter 9 (Vibration and Dynamics) and Chapter 10 (Nonlinearity in Stress Analysis) seem most appropriate. Practicing engineers as well as students may find that the book contains useful suggestions for modeling and solution strategy.

Several reviewers of the manuscript made many good suggestions. Their contributions are gratefully acknowledged. Thanks are also due to Pat Grinyer, who made it unnecessary for me to update my technical typing skills.

Robert D. Cook

Madison, Wisconsin
July 1994

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Notation

Symbols most often used in stress analysis appear in the following list. Matrices and vectors are denoted by boldface type.

LATIN SYMBOLS

A	Cross-sectional area
B	Element strain displacement matrix; $\boldsymbol{\varepsilon} = \mathbf{Bd}$
C	Constraint matrix, damping matrix
D,d	Nodal d.o.f., structure (global) and element, respectively
$\bar{\mathbf{D}}$	Amplitudes of structure (global) d.o.f. in vibration
d.o.f.	Degrees of freedom
E	Material property matrix, as in $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$
E	Elastic modulus
f	Cyclic frequency of vibration, $f = \omega/2\pi$
G	Shear modulus
I	Unit (or identity) matrix
I	Moment of inertia of cross-sectional area
J	Jacobian matrix of an isoparametric element
K,k	Stiffness matrix, structure (global) and element, respectively
L	Length
M,m	Mass matrix, structure (global) and element, respectively
N	Element shape (or interpolation) function matrix
p	Pressure
q	Distributed load along a line or on a surface
R	Vector of nodal loads applied to a structure
T	A transformation matrix
T	Temperature; also period of vibration ($T = 1/f$)
t	Thickness or time
u	Vector of displacement components, $\mathbf{u} = \{u \quad v \quad w\}$
u,v,w	Components of displacement at an arbitrary material point
V	Volume
z	Vector of scale factors of vibration modes

GREEK SYMBOLS

β_i	Generalized coordinate (amplitude of a displacement mode)
α	Coefficient of thermal expansion
$\boldsymbol{\varepsilon}$	Vector of strains; for example, $\boldsymbol{\varepsilon} = \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\}$ in the xy plane
η	An error measure, applied to the computed stress field
$\theta_x, \theta_y, \theta_z$	Rotation angles about x, y, and z axes, respectively
ν	Poisson's ratio
ξ	Damping ratio c/c_c in dynamic analyses
ξ, η, ζ	"Natural" coordinates used for isoparametric elements
ρ	Mass density or radius of curvature
$\boldsymbol{\sigma}$	Vector of stresses; for example, $\boldsymbol{\sigma} = \{\sigma_x \quad \sigma_y \quad \tau_{xy}\}$ in the xy plane
σ_e	von Mises or "effective" stress
$\boldsymbol{\phi}$	Modal matrix; its columns are vibration modes $\bar{\mathbf{D}}_i$
ω	Natural frequency of vibration (radians per second)

CHAPTER 1

Introduction

This chapter introduces concepts and procedures that are discussed in detail in subsequent chapters. The finite element (FE) analysis procedure described in Section 1.3 is used in example applications at the ends of Chapters 2, 3, 6, 7, 8, 9, and 10. Chapter 1 closes with a review of elementary matrix algebra, which is used throughout the book.

1.1 THE FINITE ELEMENT METHOD

The FE method was developed more by engineers using physical insight than by mathematicians using abstract methods. It was first applied to problems of stress analysis and has since been applied to other problems of continua. In all applications the analyst seeks to calculate a *field quantity*: in stress analysis it is the displacement field or the stress field; in thermal analysis it is the temperature field or the heat flux; in fluid flow it is the stream function or the velocity potential function; and so on. Results of greatest interest are usually peak values of either the field quantity or its gradients. The FE method is a way of getting a *numerical* solution to a *specific* problem. A FE analysis does not produce a formula as a solution, nor does it solve a class of problems. Also, the solution is approximate unless the problem is so simple that a convenient exact formula is already available.

An unsophisticated description of the FE method is that it involves cutting a structure into several elements (pieces of the structure), describing the behavior of each element in a simple way, then reconnecting elements at "nodes" as if nodes were pins or drops of glue that hold elements together (Fig. 1.1-1). This process results in a set of simultaneous algebraic equations. In stress analysis these equations are equilibrium equations of the nodes. There may be several hundred or several thousand such equations, which means that computer implementation is mandatory.

A more sophisticated description of the FE method regards it as piecewise polynomial interpolation. That is, over an element, a field quantity such as displacement is interpolated from values of the field quantity at nodes. By connecting elements together, the field quantity becomes interpolated over the entire structure in piecewise fashion, by as many polynomial expressions as there are elements. The "best" values of the field quantity at nodes are those that minimize some function such as total energy. The minimization process generates a set of simultaneous algebraic equations for values of the field quantity at nodes. Matrix symbolism for this set of equations is $\mathbf{KD} = \mathbf{R}$, where $\mathbf{D}$ is a vector of unknowns (values of the field quantity at nodes), $\mathbf{R}$ is a vector of known loads, and $\mathbf{K}$ is a matrix of known constants. In stress analysis $\mathbf{K}$ is known as a "stiffness matrix."

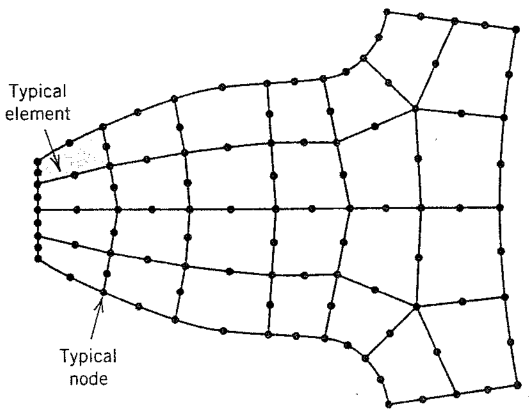

Fig. 1.1-1. A coarse-mesh, two-dimensional model of a gear tooth. All nodes and elements lie in the plane of the paper.

The power of the FE method is its versatility. The structure analyzed may have arbitrary shape, arbitrary supports, and arbitrary loads. Such generality does not exist in classical analytical methods. For example, temperature-induced stresses are usually difficult to analyze with classical methods, even when the structure geometry and the temperature field are both simple. The FE method treats thermal stresses as easily as stresses induced by mechanical load, and the temperature distribution itself can be calculated by FE.

Preprocessing and Postprocessing. The theory of FE includes matrix manipulations, numerical integration, equation solving, and other procedures carried out automatically by commercial software. The user may see only hints of these procedures as the software processes data. The user deals mainly with *preprocessing* (describing loads, supports, materials, and generating the FE mesh) and *postprocessing* (sorting output, listing, and plotting of results). In a large software package the analysis portion is accompanied by the preprocessor and postprocessor portions of the software. There also exist stand-alone pre- and postprocessors that can communicate with other large programs. Specific procedures of "pre" and "post" are different in different programs. Learning to use them is often a matter of trial, assisted by introductory notes, manuals, and on-line documentation that accompanies the software. Also, vendors of large-scale programs offer training courses. Fluency with pre- and postprocessors is helpful to the user but is unrelated to the accuracy of FE results produced. This book emphasizes how to use the FE method properly, not how to use pre- and postprocessors.

FE Method and the Typical User. The typical user of the FE method asks what kinds of elements should be used, and how many of them? Where should the mesh be fine and where may it be coarse? Can the model be simplified? How much physical detail must be represented? Is the important behavior static, dynamic, nonlinear, or what? How accurate will the answers be, and how can they be checked? One need not understand the mathematics of FE to answer these questions. However, a competent user must understand how elements behave in order to choose suitable kinds, sizes, and shapes of elements, and to guard against misinterpretations and unrealistically high expectations. A user must also realize that the FE method is a way of implementing a mathematical theory of physical behavior. Accordingly, assumptions and limitations of theory must not be violated by what we ask the software to do. In some dynamic and nonlinear analyses, algorithms by which theory is implemented must be understood, to avoid choosing an inappropriate algorithm, and to avoid interpreting results produced by algorithmic quirks or limitations as actual physical behavior. Despite all this understanding it is still easy to make mistakes in describing a problem to the computer program. Therefore it is also essential that a competent user have a good physical grasp of the problem so that errors in computed results can be detected and a judgment made as to whether the results are to be trusted or not. An analyst unable to do even a crude pencil-and-paper analysis of the problem probably does not know enough about it to attempt a solution by FE!

A Short History of FE Method. In a 1943 paper, the mathematician Courant described a piecewise polynomial solution for the torsion problem [1.1].* His work was not noticed by engineers and the procedure was impractical at the time due to the lack of digital computers. In the 1950s, work in the aircraft industry introduced the FE method to practicing engineers. A classic paper described FE work that was prompted by a need to analyze delta wings, which are too short for beam theory to be reliable [1.2]. The name "finite element" was coined in 1960 [1.3, 1.4]. By 1963 the mathematical validity of the FE method was recognized and the method was expanded from its structural beginnings to include heat transfer, groundwater flow, magnetic fields, and other areas. Large general-purpose FE software began to appear in the 1970s. By the late 1980s the software was available on microcomputers, complete with color graphics and pre- and postprocessors. By the mid-1990s roughly 40,000 papers and books about the FE method and its applications had been published.

Overview of the Remainder of the Book. Chapter 2 considers elements for bar and beam problems and discusses the mathematical structure of the FE method (the "stiffness method"). Plane problems are treated in Chapter 3. Chapter 4 discusses special methods for element formulation and linear static analysis. After studying Chapters 1 through 4 the reader should have enough background to profit from a thorough discussion of how to use the FE method properly, with attention to planning the model, detecting errors, and verifying results. This material appears in Chapter 5 and is an elaboration of Section 1.3. Chapters 6 and 7 discuss general solids, solids of revolution, plates, and shells. Temperature distribution is considered in Chapter 8, with emphasis on its use in thermal stress analysis. Vibration and other dynamic problems occupy Chapter 9. Chapter 10 is devoted to nonlinear problems and buckling. Example applications of the FE method appear near the ends of most chapters.

1.2 ELEMENTS AND NODES

Finite elements resemble fragments of the structure. Nodes appear on element boundaries and serve as connectors that fasten elements together. In Fig. 1.2-1, elements are triangular or quadrilateral areas and nodes are indicated by dots. Except for element midside nodes along *AED* and nodes at *A*, *B*, and *E*, each node acts as a connector between two or more elements. All elements that share a node have the same displacement components at that node. Lines in Fig. 1.2-1 indicate boundaries between elements. Thus we see elements with corner nodes only and elements with side nodes as well. Such a mixture of element types is neither necessary nor common but serves the present discussion.

Superficially, it appears that a FE structure can be produced by sawing the actual structure apart and then pinning it back together at nodes. Clearly, such an assemblage would be weak and unrepresentative of the actual structure because of strain concentrations at nodes, sliding of elements on one another, and even gaps that would appear be-

*Numbers in brackets indicate references listed at the back of the book.

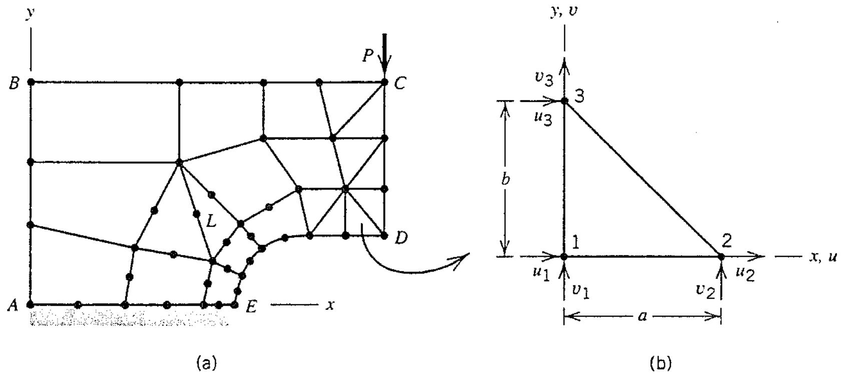

Fig. 1.2-1. (a) A flat bracket modeled by several element types (more types than would actually be used for this problem). (b) One of the elements, a "constant strain triangle". All nodes and elements lie in the plane of the paper.

tween some elements. To avoid these defects and to permit convergence toward exact results as more and more elements are used in the FE model, each element is restricted in its mode of deformation. This leads us to ask what kind of behavior can be expected of each element type. The question is answered repeatedly in subsequent chapters. For now we discuss only the following abbreviated examples of plane elements, which are discussed in more detail in Chapter 3.

Consider the plane triangular element in Fig. 1.2-1b. It does not matter that the origin of coordinates has been moved from its position in Fig. 1.2-1a. The x and y direction components of displacement of an arbitrary point within the element are given the names u and v. In the three-node triangular element each is restricted to be a linear polynomial in x and y:

$$u = \beta_1 + \beta_2 x + \beta_3 y \tag{1.2-1a}$$

$$v = \beta_4 + \beta_5 x + \beta_6 y \tag{1.2-1b}$$

where the β_i are called "generalized coordinates." They can be regarded as displacement amplitudes. As examples, in Eq. 1.2-1a, β_1 is the amplitude of rigid-body displacement, and β_2 and β_3 are amplitudes of linearly varying displacement, all in the x direction. Alternative forms of Eqs. 1.2-1 can be written by expressing the β_i in terms of nodal displacements u_1, v_1, u_2, v_2, u_3, and v_3. To do so for the element in Fig. 1.2-1b we make the following substitutions in Eqs. 1.2-1:

$$\begin{array}{llllllll} u = u_1 & \text{and} & v = v_1 & \text{at} & x = 0 & \text{and} & y = 0 \\ u = u_2 & \text{and} & v = v_2 & \text{at} & x = a & \text{and} & y = 0 \\ u = u_3 & \text{and} & v = v_3 & \text{at} & x = 0 & \text{and} & y = b \end{array} \tag{1.2-2}$$

Thus, for the element in Fig. 1.2-1b, alternative forms of Eqs. 1.2-1 are found to be

$$u = \left(1 - \frac{x}{a} - \frac{y}{b}\right)u_1 + \frac{x}{a}u_2 + \frac{y}{b}u_3 \tag{1.2-3a}$$

$$v = \left(1 - \frac{x}{a} - \frac{y}{b}\right)v_1 + \frac{x}{a}v_2 + \frac{y}{b}v_3 \tag{1.2-3b}$$

In either form, Eqs. 1.2-1 or 1.2-3, the displacement field $u = u(x, y)$ and $v = v(x, y)$ has six *degrees of freedom*, abbreviated d.o.f. That is, six quantities define the deformed configuration, namely, the six β_i in Eqs. 1.2-1 or the three u_i and three v_i in Eqs. 1.2-3. In Chapter 3 we will explain that strains are displacement gradients. Therefore

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} && \text{hence } \varepsilon_x = \beta_2 \\ \varepsilon_y &= \frac{\partial v}{\partial y} && \text{hence } \varepsilon_y = \beta_6 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} && \text{hence } \gamma_{xy} = \beta_3 + \beta_5 \end{aligned} \tag{1.2-4}$$

This three-node element is called a "constant strain triangle" because none of the strains varies over the element. This means that the element has a very limited response—it could not represent the linear strain field of pure bending, for example—but at least there will be no strain concentrations at nodes. Also, from Eqs. 1.2-3 we can conclude that element sides will remain straight after deformation. For example, set $x = 0$ to examine side 1–3 in Fig. 1.2-1b: thus u becomes linear in y and depends only on d.o.f. u_1 and u_3. The same will be true along this side in the adjacent element. Because deformed sides remain straight, elements will not gap apart or overlap when load is applied. Similarly, we can show that v along side 1-3 is linear in y and depends only on v_1 and v_3, whether we examine the element on the left or the element on the right of side 1–3. Summing up, it is possible to demonstrate that the triangular element can display constant strain states and will deform in a way that is compatible with its neighbors. The same can be demonstrated for other shapes and types of element. It can be shown that these properties allow exact results to be approached as a mesh is refined; that is, as more and more elements are used to model a structure.

Let us also consider briefly a six-node triangle, such as element L somewhat above E in Fig. 1.2-1a. It has three vertex nodes and three midside nodes. In terms of generalized coordinates β_i, its displacement field is

$$\begin{aligned} u &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 \\ v &= \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2 \end{aligned} \tag{1.2-5}$$

Deformed shapes of sides can be straight or parabolic. Some tedious algebra shows that the deformed shape of a side depends on d.o.f. of nodes attached to that side but does not depend on d.o.f. of nodes *not* attached to that side. Accordingly, the element will be compatible with its neighbors because adjacent elements share the same nodes and d.o.f. along a common side. By applying the differentiation used in Eqs. 1.2-4, we see that the six-node element contains constant and linear terms in its strain field. Therefore this element can model constant strain states and also linear strain states that arise in pure bending. Clearly, it is a more competent element than the constant strain triangle. It is also more complicated, which suggests another choice faced by the user of FE: Is it better to