FINITE ELEMENTS IN PLASTICITY:

Theory and Practice

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Preface

The purpose of this text is to present and demonstrate the use of finite element based methods for the solution of problems involving plasticity. As well as the conventional quasi-static incremental theory of plasticity, attention is given to the slow transient phenomenon of elasto-viscoplastic behaviour and also to dynamic transient problems. We make no pretence that the text provides a complete treatment of any of these topics but rather we see it as an attempt to present numerical solution techniques, which have been well tried and tested, for selected important areas of application.

In our earlier books on finite elements we have concentrated on linear applications. Here we attempt the much more daunting task of introducing, in detail, the use of finite elements for solving problems in which plasticity effects are present. To our knowledge it is the first such book. Our main idea is to present the theory and detailed algorithms in the form of modular routines written in FORTRAN which can be linked together to form 13 finite element plasticity programs.

Writing this book has been in itself, rather like solving a nonlinear finite element problem. We have gone through many iterations and we hope that we have now converged to a reasonable 'solution'. As in many real engineering situations our convergence criterion has been influenced by a deadline. In our case the deadline was largely self-imposed as we have already been engaged on this project for more than three years. We do not believe our solution to be unique or in any sense optimal. We merely offer it to fill a gap in the existing literature.

The text is arranged in three main parts. Part I is devoted to one-dimensional problems. These relatively simple applications are possibly the most important in the book; since all the essential features of nonlinear finite element analysis are immediately recognisable without the distractions and complications that are present in general continuum problems. Part II deals with the two-dimensional applications of plane stress/strain and axi-symmetric continua and plate bending problems. Finally in Part III we present some dynamic transient applications and briefly describe some further developments.

All of the programs presented in this text have been specially written by the authors. In the development of the subroutines for the solution algorithms described, a conflict inevitably arose between computational efficiency and clarity of coding. Whatever sacrifices have been made have been biased towards satisfying the latter condition. However, we believe that the codes presented are both reasonably efficient and flexible and have potential usage in commercial as well as teaching and research environments. A total of 132 subroutines are presented which amount to more than 8,000 statements. The 13 assembled programs comprise approximately 20,000 statements. To aid readers wishing to implement the programs a magnetic tape of the computer codes together with the test input data listed in Appendix IV is available from the publishers. Although every attempt has been made to verify the programs, no responsibility can be accepted for their performance in practice.

A further feature of the book is that each chapter contains several exercises for further study.

We are indebted to many people for their direct or indirect assistance in the preparation of this text. This preface would not be complete without an acknowledgment of this debt and a record of our gratitude to the following: To Professor O. C. Zienkiewicz for his pioneering work and stimulating influence. To Professor G. C. Nayak whose work on numerical analysis of plasticity problems has significantly influenced the present text. To Dr. I. C. Cormeau whose thesis on viscoplasticity has been an invaluable source of information. To Professor K. J. Bathe for permission to use the profile equation solver included in Chapter 11. To N. Bicanic, D. K. Paul, H. H. Abdel Rahman and M. M. Huq for their generous assistance in the preparation of several chapters. To our colleagues and former research workers in the Department of Civil Engineering, University College of Swansea for helpful discussions and suggestions. To E. S. Caldis for his care in preparing annotated computer listings and, finally, to Mrs. M. J. Davies for her skill and patience in typing the manuscript.

D. R. J. OWEN
E. HINTON

Swansea, May 1980

Part I

Chapter 1
Introduction

1.1 Introductory remarks

The finite element method is now firmly accepted as a most powerful general technique for the numerical solution of a variety of problems encountered in engineering. Applications range from the stress analysis of solids to the solution of acoustical, neutron physics and fluid dynamics problems. Indeed the finite element process is now established as a general numerical method for the solution of partial differential equation systems, subject to known boundary and/or initial conditions.

For linear analysis, at least, the technique is widely employed as a design tool. Similar acceptance for nonlinear situations is dependent on two major factors. Firstly, in view of the increased numerical operations associated with nonlinear problems, considerable computing power is required. Developments in the last decade or so have ensured that high-speed digital computers which meet this need are now available and present indications are that reductions in unit computing costs will continue. Secondly, before the finite element method can be used in design, the accuracy of any proposed solution technique must be proven. The development of improved element characteristics and more efficient nonlinear solution algorithms and the experience gained in their application to engineering problems have ensured that nonlinear finite element analyses can now be performed with some confidence. Hence barriers to the common use of nonlinear finite element techniques are being rapidly removed and the process is already economically acceptable for selected industrial applications.

1.2 Aims and layout

The object of this book is to describe in detail the application of the finite element method to the solution of materially nonlinear engineering analysis problems. Unlike other texts on linear and nonlinear finite element analysis[1–4] which have dealt predominantly with theoretical aspects, this book is intended to be more practical and therefore focuses attention on the *computer implementation* of nonlinear finite element schemes.

Nonlinearities arise in engineering situations from several sources. For example a nonlinear material response can result from elasto-plastic material behaviour or from hyperelastic effects of some form. Additionally nonlinear characteristics can be associated with temporal effects such as viscoplastic behaviour or dynamic transient phenomena. Each of these nonlinearities may occur in a variety of structural types such as two- or three-dimensional solids, frames, plates or shells. Therefore it becomes clear that a textbook dealing with nonlinear finite element programming must at least be restricted to selected topics. For this reason three classes of problems will be examined in depth in the three parts of this text.

Part I: One-dimensional materially nonlinear problems. All the essential features of a nonlinear finite element solution can be described in relation to one-dimensional models. The applications considered are:

- Nonlinear quasi-harmonic problems
- Nonlinear elastic situations
- Elasto-plastic behaviour of axial bar systems
- Time dependent elasto-viscoplastic analysis of bar systems
- Elasto-plastic beam bending

Part II: Two-dimensional materially nonlinear problems. In this part the ideas developed in Part I are extended to continuum problems. The following applications are presented:

- Elasto-plastic analysis of plane stress, plane strain and axi-symmetric solids
- Time dependent elasto-viscoplastic analysis of plane stress, plane strain and axisymmetric solids
- Elasto-plastic plate bending problems

Part III: Nonlinear transient dynamic problems. In this time-dependent class of problems inertia effects are included in the analysis. In this part, the following topics are considered:

- Elasto-plastic and geometrically nonlinear material behaviour
- Explicit and implicit time integration schemes
- Combined explicit/implicit algorithms

It should be pointed out that several different programming options are open for solution of the above problems and the methods presented in this text are the ones which are physically the most clear and which experience indicates give reliable results for a wide range of applications. An important feature of this text is the step-by-step development of thirteen finite element programs to deal with the above problems.

For the one-dimensional applications considered in Part I, only a 2-node element with linear displacement variation between nodes is considered. This allows the basic steps of a nonlinear finite element analysis to be presented without unnecessary distractions. In Parts II and III of the text, where two-dimensional continuum and plate bending problems are considered, isoparametric elements are exclusively employed. In particular, a

4-node linear element and 8- and 9-node quadratic versions are used. These elements are illustrated in Fig. 1.1 and are extremely versatile, good performers which have been well tried and tested in both linear and nonlinear situations. A typical elasto-plastic application using 8-node isoparametric elements is shown in Fig. 1.2 where the incremental loading of a notched beam is illustrated. The progressive development of plastic zones with increasing load levels are compared for a Tresca and Von Mises yield criterion.

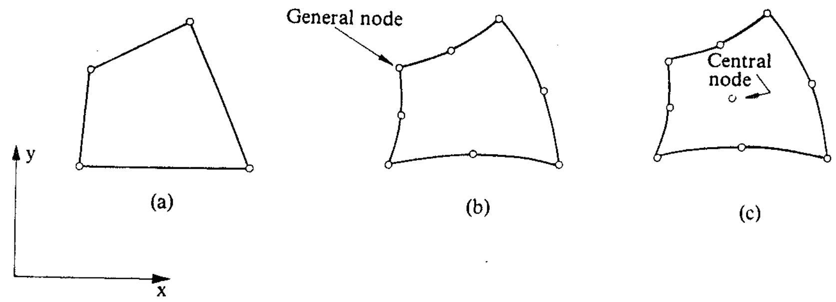

Fig. 1.1 The two-dimensional isoparametric elements employed in the text: (a) Linear 4-node; (b) Serendipity 8-node; (c) Lagrangian 9-node.

The layout of the book will now be briefly described. The remainder of Chapter 1 discusses the basic notation and style adopted in program presentation.

Chapter 2 discusses the general nonlinear problem and some solution techniques are outlined. For the one-dimensional applications to be considered, basic theoretical expressions are developed in a form suitable for numerical solution.

In Chapter 3, the solution techniques presented in Chapter 2 are programmed in FORTRAN and numerical examples are solved for each separate application.

Chapter 4 is devoted to one-dimensional elasto-viscoplastic problems. The basic theory for this time-dependent phenomenon is first presented. The process is then coded and the program used to solve some numerical examples.

In Chapter 5 elasto-plastic beam bending is considered. This topic forms a bridge between uniaxial and continuum applications since now more than one degree of freedom exists at each nodal point. Some measure of continuum behaviour is also introduced since a layered approach is used to trace the development of plasticity through the cross-section of the beam.

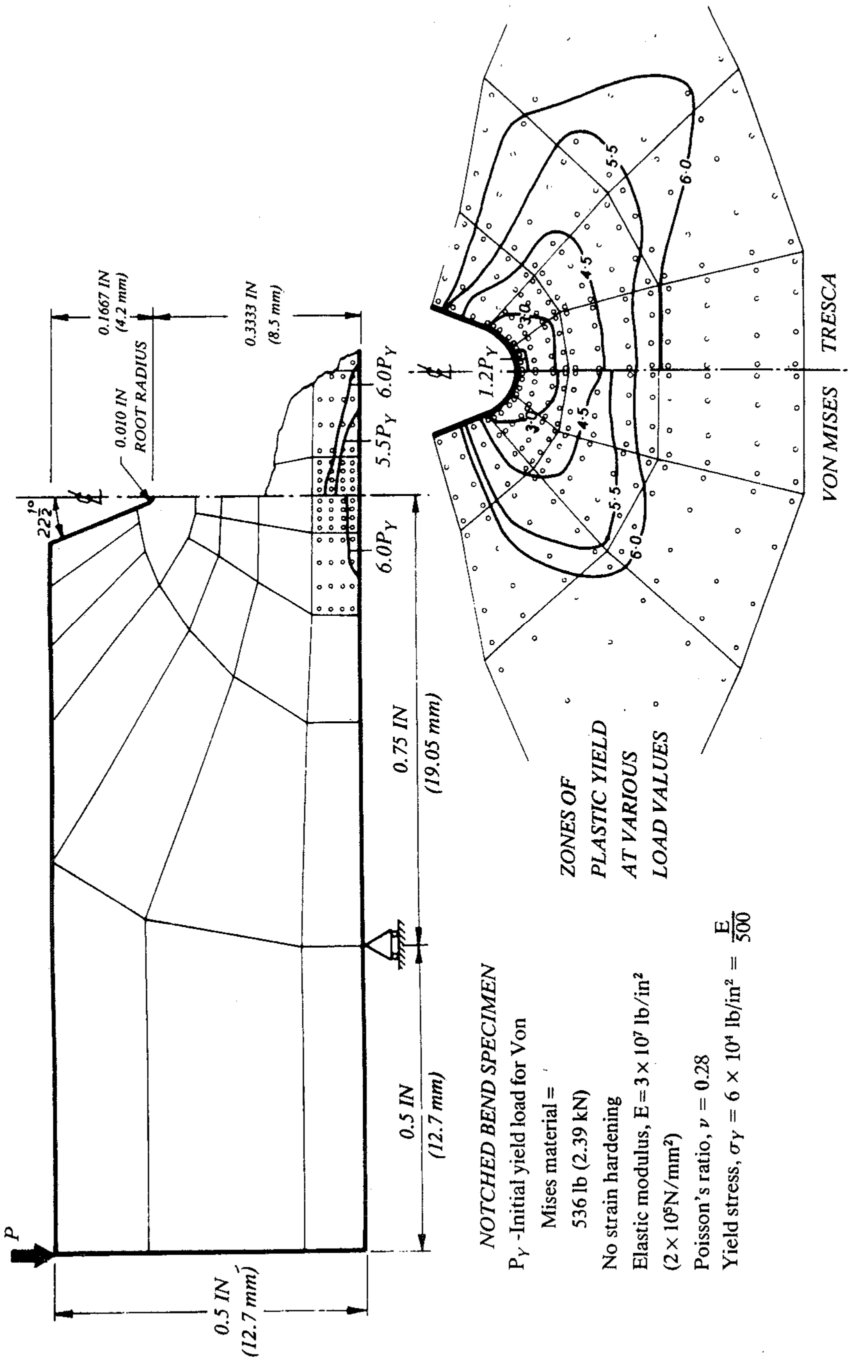

Fig. 1.2 Elasto-plastic analysis of a notched beam under bending showing plastic zone distributions for both a Von Mises and a Tresca yield criterion.

Chapter 6 forms an introduction to two-dimensional continuum problems. The basic theory for two-dimensional isoparametric elements is presented and some standard subroutines required for applications described in later chapters are listed. These include routines which perform some standard linear elastic operations, such as nodal load generation, equation solution, etc., as well as nonlinear subroutines common to more than one application.

Two-dimensional elasto-plastic problems are considered in Chapter 7. Basic theoretical expressions for a general continuum are first reviewed, and manipulated into forms convenient for numerical analysis. Particular expressions for plane stress/strain and axisymmetric situations are then developed and coded. Four different yield criteria are employed. The Tresca and Von Mises laws which closely approximate metal plasticity behaviour are considered and the Mohr–Coulomb and Drucker–Prager criteria, which are applicable to concrete, rocks and soil are presented.

Chapter 8 is concerned with the transient phenomenon of elasto-viscoplasticity where again the situations of plane stress/strain and axial symmetry are considered. Both explicit and implicit time integration schemes are presented and the four yield criteria considered in Chapter 7 are employed. The FORTRAN program developed is illustrated by application to some numerical examples.

Elasto-plastic plate bending problems are discussed in Chapter 9. The basic theoretical expressions are presented in a form suitable for numerical analysis with both a layered and nonlayered approach to plastification through the plate thickness being considered. Treatment in this chapter is limited to the Tresca and Von Mises yield conditions.

Chapters 10 and 11 deal with the transient dynamic analysis of two-dimensional continua. In this application inertia effects are included in the computation and problems such as blast loading and seismic phenomena are considered. Nonlinear effects due to both elasto-plastic material behaviour and gross geometric deformations are included. Both explicit and implicit techniques are employed for the time integration of the equations of motion as well as a combined implicit/explicit algorithm. The computer codes developed are applied to the solution of some practical problems.

Finally in Chapter 12 further aspects of nonlinear material behaviour are discussed. Alternative solution techniques and material models are referred to and some additional fields of application indicated.

Three appendices are included which contain user instructions for the computer programs described throughout the text. Appendices I and II provide user instructions for one-dimensional and two-dimensional continuum problems respectively. A user's guide for transient dynamic problems is provided in Appendix III. Finally in Appendix IV sample input data and lineprinter output are provided for both one- and two-dimensional applications.

1.3 Program structure

1.3.1 Introduction

This section describes the main features of the computer programs to be developed later in the book. A modular approach is adopted, in that separate subroutines are employed to perform the various operations required in a nonlinear finite element analysis. Generally each program consists of 9 modules, each with a distinct operational function. Each module in turn is composed of one or more subroutines relevant only to its own needs and, in some cases, of subroutines which are common to several modules. Control of the modules is held by the main or master segment.

The modules, shown schematically in Fig. 1.3, are described in relation to their general functions as follows:

1. *Initialisation or zeroing module*—this is the first module entered and its function is to initialise to zero various vectors and matrices at the beginning of the solution process.
2. *Data input and checking module*—this is the second module entered. It handles input data defining the geometry, boundary conditions and material properties. This data is checked using diagnostic routines and if errors occur they are flagged and the remainder of the input data is printed out before the program is terminated. For isoparametric elements, Gaussian integration constants and mid-side nodal coordinates for straight-sided elements are also evaluated in this section. Once used this module is not needed again.
3. *Loading module*—this module organises the calculation of nodal forces due to the various forms of loading for two-dimensional application. These include pressure, gravity and concentrated loadings.
4. *Load incrementing module*—Any materially nonlinear finite element solution must proceed on an incremental basis. Therefore the function of this section is to control the incrementing of the applied loads evaluated by the loading module. It also ensures that any specified displacement values are also incrementally applied.
5. *Stiffness module*—this is the next module entered and organises the evaluation of the stiffness matrix for each element. The stiffness matrices are stored on disc and ordered in the sequence required for equation assembly and reduction.
6. *Solution module*—the general purpose of this routine is to assemble, reduce and solve the governing set of simultaneous equations to give the nodal displacements and force reactions at restrained nodal points.
7. *Residual force module*—the function of this module is to calculate the residual or 'out of balance' nodal forces at each stage of the analysis.
8. *Convergence module*—in this module the convergence of the nonlinear solution is checked against criteria given in later chapters.

9. *Output module*—this module organises the output of the requested quantities.

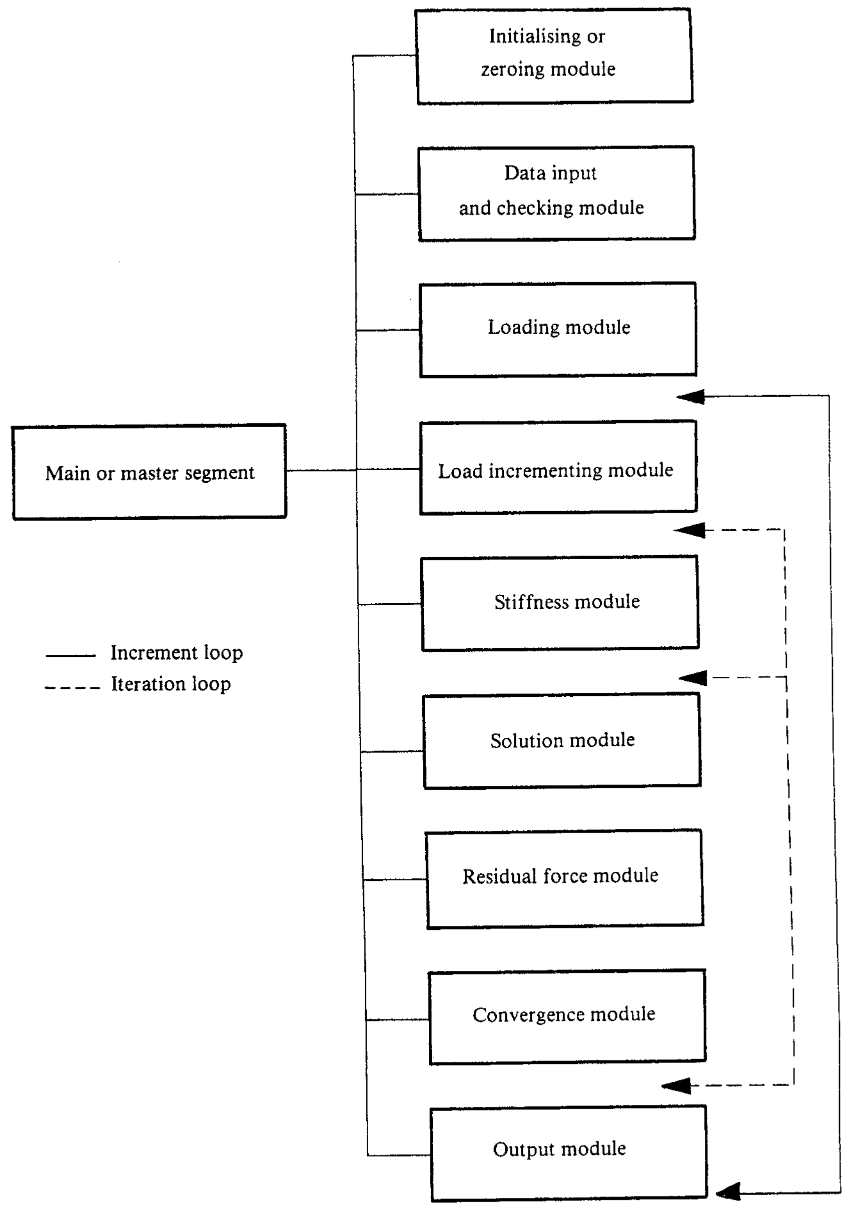

Fig. 1.3 Program modules for nonlinear solution codes.

The main purpose of the main or master segment is to call the above modules and to control the load increments and iteration procedure according to the solution algorithm being employed and the convergence rate of the solution process.

1.3.2 Programming notation

In the programs presented in this text an attempt has been made to name variables in a logical manner. By choosing descriptive names, the use of many of the variables becomes self-apparent, thus assisting the reader in the task of program assimilation. All variable names are chosen to be 5 characters in length; this occasionally causes a little difficulty in abbreviation but has an advantage with regard to neatness of program presentation. For example, the following names will be employed.

NMATS	The Number of different MATerialS
PROPS ( )	The array of material PROPertieS
NEVAB	The Number of Element VAriaBles
NNODE	The Number of NODes per Element
NDOFN	The Number of Degrees Of Freedom per Node

Furthermore a 'common root' principle will be adopted; where a single basic variable name is employed with different prefixes depending on its usage in the program. In particular:

i) Prefix I, J or L will be used to indicate a DO loop variable
ii) Prefix K will indicate a counter
iii) Prefix M will indicate a maximum value
iv) Prefix N will indicate a given number

For example IPOIN, NPOIN, MPOIN will indicate respectively a particular nodal point, the number of nodal points in the problem and the maximum permissible number of nodal points in the program.

Similarly, any DO loop will be of the general form

```
  KEVAB=0
  DO 1 INODE=1, NNODE
  DO 1 IDOFN=1, NDOFN
1 KEVAB=KEVAB+1
```

which indicates that the outer and inner DO loop indices range respectively over the number of nodes per element and the number of degrees of freedom per node. The prefix K is employed in KEVAB to indicate a counter over the number of element variables, NEVAB.

All programming is undertaken in standard FORTRAN IV. A listing is presented for all subroutines described in this text and detailed notes on each group of statements are provided. Comment cards have also been used to assist in the understanding of the programs.

1.4 References

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Chapter 2
One-dimensional nonlinear problems

2.1 Introduction

Several classes of nonlinear problems of interest in many branches of science and engineering can be reduced to the solution of a system of simultaneous equations in which the equation coefficients are dependent on some function of the prime variables.[1] In this chapter some basic techniques for the numerical solution of such problems are examined. In order to introduce the essential details of the solution processes as simply as possible, the applications will be restricted to one-dimensional situations. In particular, elasto-plasticity, nonlinear elasticity problems and systems governed by a nonlinear quasi-harmonic equation will be considered. In each case a computer program will be developed and its use illustrated by application to simple problems. The aim of this chapter is to prepare the reader for the more comprehensive two-dimensional treatment of these topics which will be undertaken in Chapters 6–9. Indeed, all the essential features of nonlinear finite element analysis detailed in these later chapters will be recognisable from the simple treatment considered here. It should be emphasised that the subroutines developed in this chapter will *not* be used in the main finite element programs discussed in Parts II and III.

2.2 Basic numerical solution processes for nonlinear problems

The use of finite element discretisation in a large class of nonlinear problems results in a system of simultaneous equations of the form

$$H\varphi + f = 0, \tag{2.1}$$

in which φ is the vector of the basic unknowns, f is the vector of applied 'loads' and H is the assembled 'stiffness' matrix. For structural applications, the terms 'load' and 'stiffness' are directly applicable, but for other situations the interpretation of these quantities varies according to the physical problem under consideration.

If the coefficients of the matrix H depend on the unknowns φ or their derivatives, the problem clearly becomes nonlinear. In this case, direct solution of equation system (2.1) is generally impossible and an iterative scheme must be adopted. Many options remain open for the iterative sequence to be employed. Some of the most generally applicable methods available will now be outlined.

2.2.1 Method of direct iteration (or successive approximations)

In this approach[2] successive solutions are performed, in each of which the previous solution for the unknowns φ is used to predict the current values of the coefficient matrix $H(\varphi)$. Rewriting (2.1) as

$$\varphi = -[H(\varphi)]^{-1}f, \tag{2.2}$$

then the iterative process yields the $(r+1)^{\text{th}}$ approximation to be

$$\varphi^{r+1} = -[H(\varphi^r)]^{-1}f. \tag{2.3}$$

If the process is convergent then in the limit as r tends to infinity φ^r tends to the true solution.

It is seen from (2.3) that it is necessary to recalculate the 'stiffness' matrix H for each iteration. To commence the process, an initial guess for the unknown φ is required in order to calculate H. Generally a value of φ^0 based on the solution for an average material property throughout the region is found to be satisfactory. If the nonlinearity of the material properties is very marked at certain values of φ, an approximate prescription of the field variable at all nodes may be necessary.

For practical purposes, the iterative process is deemed to have converged when some measure (usually a norm of the nodal unknowns) of the change in the unknown φ between successive iterations has become tolerably small. The process is illustrated diagrammatically for a single variable in Figs 2.1 and 2.2, in which case the matrix H and vector φ reduce to the scalar equivalents H and ϕ. The assumed dependence of H on ϕ is a basic problem function which must be prescribed before solution can commence. This material property is included in Figs 2.1 and 2.2 and, for convenience, the relationship between $H(\phi).\phi$ and ϕ is prescribed rather than the $H(\phi)-\phi$ dependence. Figure 2.1 shows the convergence paths for initial trial values, ϕ^0, which are below and above the true solution, ϕ_T, and for a convex $H-\phi$ relation. From the initial trial value, ϕ^0, the corresponding value of H is immediately given from the prescribed $H(\phi).\phi-\phi$ relationship, to be H^0. Equation (2.3) is then solved to give ϕ^1. The value of H corresponding to ϕ^1 is then determined from the $H(\phi).\phi-\phi$ relationship and (2.3) then resolved to obtain ϕ^2. This cycling process is continued until ϕ^{n-1} and ϕ^n are deemed to be sufficiently close, indicating that convergence has occurred. The quantity H^r is represented by the slope of the secant to the $H-\phi$ curve and decreases with increasing values of ϕ. Both the high and low initial trial solutions produce monotonic convergence paths. Figure 2.2 shows the unsuitability of the method for problems with a concave $H-\phi$ relationship. Both low and high initial trial solutions produce convergence paths which oscillate around the true solution. Although the solution converges for the

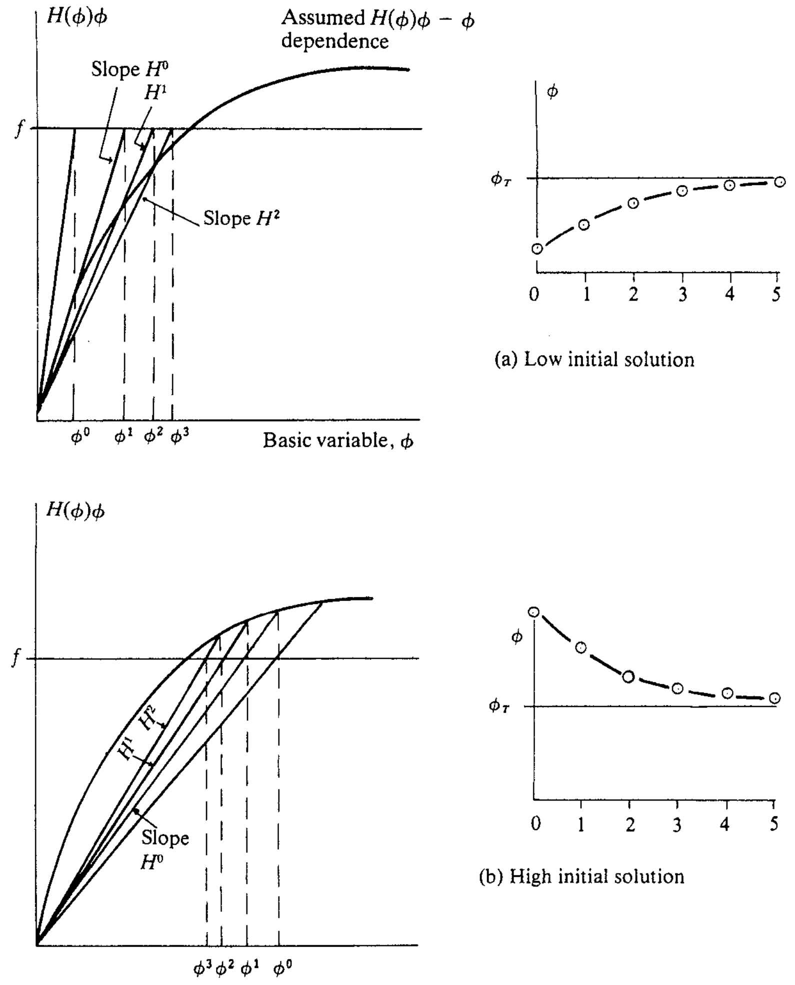

Fig. 2.1 Direct iteration method for a single variable problem—convex H-ϕ relation.

single variable case, in multi-degree of freedom problems the coupling of stiffness terms is likely to lead to instability of the iterative process. A disadvantage of the direct iteration method is that convergence of the solution scheme is not guaranteed and cannot be predicted at the initial solution stage.

2.2.2 The Newton–Raphson method

During any step of an iterative process of solution, (2.1) will not be satisfied unless convergence has occurred. A system of *residual forces* can be assumed

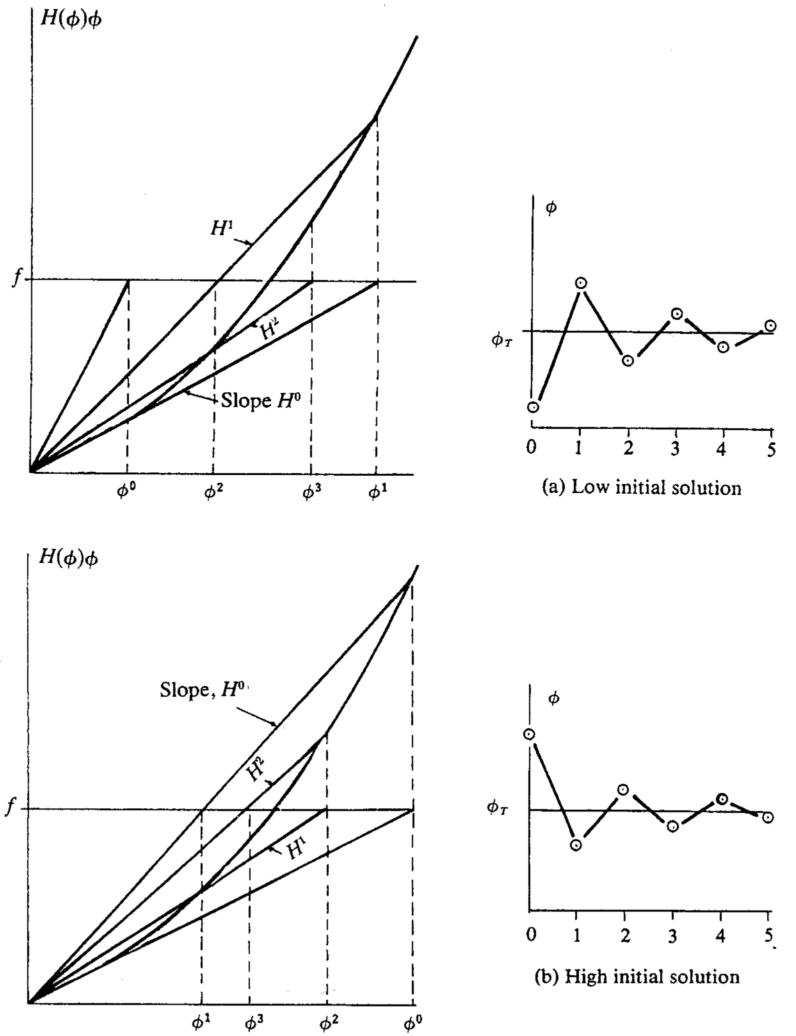

Fig. 2.2 Direct iteration method for a single variable problem—concave H-ϕ relation.

to exist, so that

$$\psi = H\varphi + f \neq 0. \tag{2.4}$$

These residual forces ψ can be interpreted as a measure of the departure of (2.1) from equilibrium. Since H is a function of φ and possibly its derivatives, then at any stage of the process, $\psi = \psi(\varphi)$.

If the true solution to the problem exists at $\varphi^r + \Delta\varphi^r$ then the Newton–Raphson approximation[2] for the general term of the residual force vector, ψ^r corresponding to solution at φ^r is

$$\psi_i^r = -\sum_{j=1}^{N} \Delta\phi_j^r \left(\frac{\partial\psi_i}{\partial\phi_j}\right)^r, \tag{2.5}$$

in which N is the total number of variables in the system and the superscript r denotes the r^{th} approximation to the true solution. Substituting for ψ_i from (2.4), the complete expression for all the residual components can be written in matrix form as

$$\psi(\varphi^r) = -J(\varphi^r)\Delta\varphi^r. \tag{2.6}$$

in which a typical term of the Jacobian matrix J is

$$J_{ij} = \left(\frac{\partial\psi_i}{\partial\phi_j}\right)^r = h_{ij}{}^r + \sum_{k=1}^{m}\left(\frac{\partial h_{ik}}{\partial\phi_j}\right)^r \phi_k{}^r, \tag{2.7}$$

where h_{ij} is the general term of matrix H. The last term in (2.7) gives rise to nonsymmetric terms in the Jacobian matrix. If these nonsymmetric terms are neglected in order to maintain symmetry, then substitution of (2.7) in (2.6) results in

$$H(\varphi^r).\Delta\varphi^r = -\psi(\varphi^r). \tag{2.8}$$

Or since

$$\Delta\varphi^r = \varphi^{r+1} - \varphi^r, \tag{2.9}$$

equation (2.8) reduces, on use of (2.4), to

$$H(\varphi^r).\varphi^{r+1} + f = 0. \tag{2.10}$$

This equation is identical to equation (2.3), Section 2.2.1, which governs the method of direct iteration. Therefore in order to achieve the better convergence rate associated with the Newton–Raphson process it is essential that the unsymmetric terms in J be retained.

The explicit form of the nonlinear terms in (2.7) will clearly depend on the way in which the stiffness matrix coefficients, h_{ij}, depend on the unknowns, φ. The terms of the Jacobian matrix, given in (2.7), can be assembled to give the general expression

$$J(\varphi) = H(\varphi) + H'(\varphi), \tag{2.11}$$

where the last term contains the unsymmetric terms only. The Newton–Raphson process can be finally written, using (2.6) and (2.11), in the form

$$\Delta\varphi^r = -[J(\varphi^r)]^{-1}.\psi(\varphi^r) = -[H(\varphi^r) + H'(\varphi^r)]^{-1}\psi(\varphi^r). \tag{2.12}$$

This allows the correction to the vector of unknowns φ to be obtained from the residual force vector ψ for any iteration. Again an iterative approach must be followed, with the vector of unknowns φ being corrected at each stage according to (2.12) until convergence of the process is deemed to have occurred. The technique is illustrated schematically in Figs 2.3 and 2.4 for

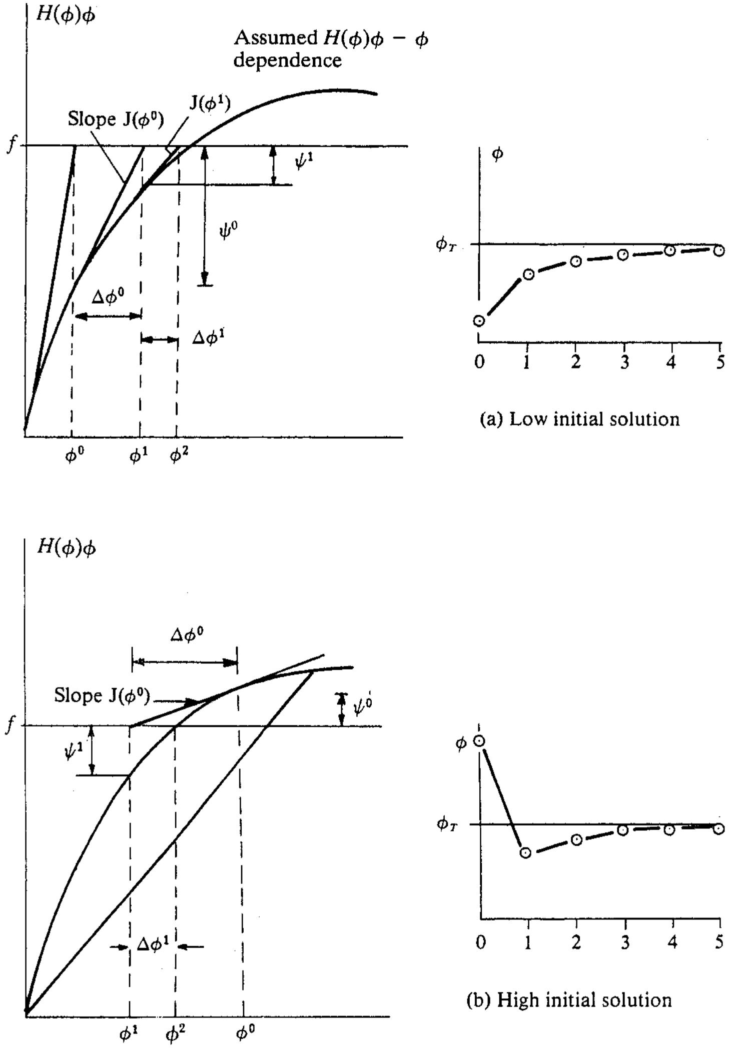

Fig. 2.3 The Newton–Raphson method for a single variable problem—convex H–ϕ relation.

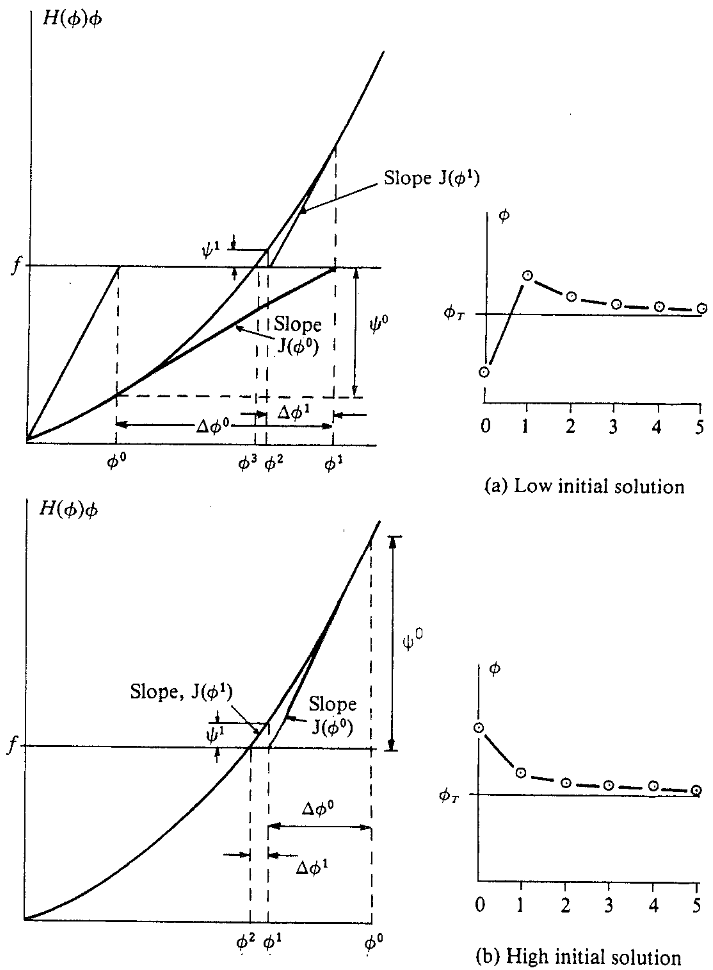

Fig. 2.4 The Newton–Raphson method for a single variable problem—concave H–ϕ relation.

a single variable situation. Solution to the nonlinear problem will be achieved when the residual force ψ vanishes, since this term directly measures the lack of equilibrium of the governing equation as indicated in (2.4). A trial value φ^0 of the basic unknown is assumed and the material stiffness associated with this value calculated according to the prescribed $H-\varphi$ relationship.