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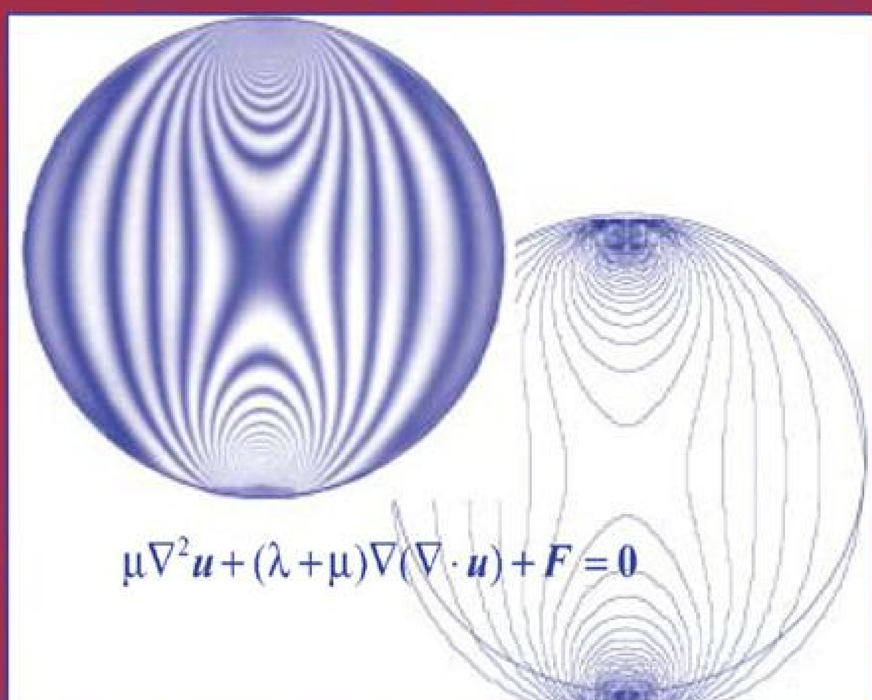
Elasticity

Theory, Applications, and Numerics



Martin H. Sadd

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Preface

This text is an outgrowth of lecture notes that I have used in teaching a two-course sequence in theory of elasticity. Part I of the text is designed primarily for the first course, normally taken by beginning graduate students from a variety of engineering disciplines. The purpose of the first course is to introduce students to theory and formulation and to present solutions to some basic problems. In this fashion students see how and why the more fundamental elasticity model of deformation should replace elementary strength of materials analysis. The first course also provides the foundation for more advanced study in related areas of solid mechanics. More advanced material included in Part II has normally been used for a second course taken by second- and third-year students. However, certain portions of the second part could be easily integrated into the first course.

So what is the justification of my entry of another text in the elasticity field? For many years, I have taught this material at several U.S. engineering schools, related industries, and a government agency. During this time, basic theory has remained much the same; however, changes in problem solving emphasis, research applications, numerical/computational methods, and engineering education pedagogy have created needs for new approaches to the subject. The author has found that current textbook titles commonly lack a concise and organized presentation of theory, proper format for educational use, significant applications in contemporary areas, and a numerical interface to help understand and develop solutions.

The elasticity presentation in this book reflects the words used in the title—*Theory, Applications* and *Numerics.* Because *theory* provides the fundamental cornerstone of this field, it is important to first provide a sound theoretical development of elasticity with sufficient rigor to give students a good foundation for the development of solutions to a wide class of problems. The theoretical development is done in an organized and concise manner in order to not lose the attention of the less-mathematically inclined students or the focus of applications. With a primary goal of solving problems of engineering interest, the text offers numerous *applications* in contemporary areas, including anisotropic composite and functionally graded materials, fracture mechanics, micromechanics modeling, thermoelastic problems, and computational finite and boundary element methods. Numerous solved example problems and exercises are included in all chapters. What is perhaps the most unique aspect of the text is its integrated use of *numerics.* By taking the approach that applications of theory need to be observed through calculation and graphical display, numerics is accomplished through the use of MATLAB, one of the most popular engineering software packages. This software is used throughout the text for applications such as: stress and strain transformation, evaluation and plotting of stress and displacement distributions, finite element calculations, and making comparisons between strength of materials, and analytical and numerical elasticity solutions. With numerical and graphical evaluations, application problems become more interesting and useful for student learning.

Text Contents

The book is divided into two main parts; the first emphasizes formulation details and elementary applications. Chapter 1 provides a mathematical background for the formulation of elasticity through a review of scalar, vector, and tensor field theory. Cartesian index tensor notation is introduced and is used throughout the formulation sections of the book. Chapter 2 covers the analysis of strain and displacement within the context of small deformation theory. The concept of strain compatibility is also presented in this chapter. Forces, stresses, and equilibrium are developed in Chapter 3. Linear elastic material behavior leading to the generalized Hook's law is discussed in Chapter 4. This chapter also includes brief discussions on non-homogeneous, anisotropic, and thermoelastic constitutive forms. Later chapters more fully investigate anisotropic and thermoelastic materials. Chapter 5 collects the previously derived equations and formulates the basic boundary value problems of elasticity theory. Displacement and stress formulations are made and general solution strategies are presented. This is an important chapter for students to put the theory together. Chapter 6 presents strain energy and related principles including the reciprocal theorem, virtual work, and minimum potential and complimentary energy. Two-dimensional formulations of plane strain, plane stress, and anti-plane strain are given in Chapter 7. An extensive set of solutions for specific two-dimensional problems are then presented in Chapter 8, and numerous MATLAB applications are used to demonstrate the results. Analytical solutions are continued in Chapter 9 for the Saint-Venant extension, torsion, and flexure problems. The material in Part I provides the core for a sound one-semester beginning course in elasticity developed in a logical and orderly manner. Selected portions of the second part of this book could also be incorporated in such a beginning course.

Part II of the text continues the study into more advanced topics normally covered in a second course on elasticity. The powerful method of complex variables for the plane problem is presented in Chapter 10, and several applications to fracture mechanics are given. Chapter 11 extends the previous isotropic theory into the behavior of anisotropic solids with emphasis for composite materials. This is an important application, and, again, examples related to fracture mechanics are provided. An introduction to thermoelasticity is developed in Chapter 12, and several specific application problems are discussed, including stress concentration and crack problems. Potential methods including both displacement potentials and stress functions are presented in Chapter 13. These methods are used to develop several three-dimensional elasticity solutions. Chapter 14 presents a unique collection of applications of elasticity to problems involving micromechanics modeling. Included in this chapter are applications for dislocation modeling, singular stress states, solids with distributed cracks, and micropolar, distributed voids, and doublet mechanics theories. The final Chapter 15 provides a brief introduction to the powerful numerical methods of finite and boundary element techniques. Although only two-dimensional theory is developed, the numerical results in the example problems provide interesting comparisons with previously generated analytical solutions from earlier chapters.

The Subject

Elasticity is an elegant and fascinating subject that deals with determination of the stress, strain, and displacement distribution in an elastic solid under the influence of external forces. Following the usual assumptions of linear, small-deformation theory, the formulation establishes a mathematical model that allows solutions to problems that have applications in many engineering and scientific fields. Civil engineering applications include important contributions to stress and deflection analysis of structures including rods, beams, plates, and shells. Additional applications lie in geomechanics involving the stresses in such materials as soil, rock, concrete, and asphalt. Mechanical engineering uses elasticity in numerous problems in analysis and design of machine elements. Such applications include general stress analysis, contact stresses, thermal stress analysis, fracture mechanics, and fatigue. Materials engineering uses elasticity to determine the stress fields in crystalline solids, around dislocations and in materials with microstructure. Applications in aeronautical and aerospace engineering include stress, fracture, and fatigue analysis in aerostructures. The subject also provides the basis for more advanced work in inelastic material behavior including plasticity and viscoelasticity, and to the study of computational stress analysis employing finite and boundary element methods.

Elasticity theory establishes a mathematical model of the deformation problem, and this requires mathematical knowledge to understand the formulation and solution procedures. Governing partial differential field equations are developed using basic principles of continuum mechanics commonly formulated in vector and tensor language. Techniques used to solve these field equations can encompass Fourier methods, variational calculus, integral transforms, complex variables, potential theory, finite differences, finite elements, etc. In order to prepare students for this subject, the text provides reviews of many mathematical topics, and additional references are given for further study. It is important that students are adequately prepared for the theoretical developments, or else they will not be able to understand necessary formulation details. Of course with emphasis on applications, we will concentrate on theory that is most useful for problem solution.

The concept of the elastic force-deformation relation was first proposed by Robert Hooke in 1678. However, the major formulation of the mathematical theory of elasticity was not developed until the 19th century. In 1821 Navier presented his investigations on the general equations of equilibrium, and this was quickly followed by Cauchy who studied the basic elasticity equations and developed the notation of stress at a point. A long list of prominent scientists and mathematicians continued development of the theory including the Bernoulli's, Lord Kelvin, Poisson, Lamé, Green, Saint-Venant, Betti, Airy, Kirchhoff, Lord Rayleigh, Love, Timoshenko, Kolosoff, Muskhelishvilli, and others. During the two decades after World War II, elasticity research produced a large amount of analytical solutions to specific problems of engineering interest. The 1970s and 1980s included considerable work on numerical methods using finite and boundary element theory. Also, during this period, elasticity applications were directed at anisotropic materials for applications to composites. Most recently, elasticity has been used in micromechanical modeling of materials with internal defects or heterogeneity. The rebirth of modern continuum mechanics in the 1960s led to a review of the foundations of elasticity and has established a rational place for the theory within the general framework. Historical details may be found in the texts by: Todhunter and Pearson, *History of the Theory of Elasticity;* Love, *A Treatise on the Mathematical Theory of Elasticity;* and Timoshenko, *A History of Strength of Materials.*

Exercises and Web Support

Of special note in regard to this text is the use of exercises and the publisher's web site, *www.books.elsevier.com*. Numerous exercises are provided at the end of each chapter for homework assignment to engage students with the subject matter. These exercises also provide an ideal tool for the instructor to present additional application examples during class lectures. Many places in the text make reference to specific exercises that work out details to a particular problem. Exercises marked with an asterisk (*) indicate problems requiring numerical and plotting methods using the suggested MATLAB software. Solutions to all exercises are provided on-line at the publisher's web site, thereby providing instructors with considerable help in deciding on problems to be assigned for homework and those to be discussed in class. In addition, downloadable MATLAB software is also available to aid both students and instructors in developing codes for their own particular use, thereby allowing easy integration of the numerics.

Feedback

The author is keenly interested in continual improvement of engineering education and strongly welcomes feedback from users of this text. Please feel free to send comments concerning suggested improvements or corrections via surface or e-mail (sadd@egr.uri.edu). It is likely that such feedback will be shared with text user community via the publisher's web site.

Acknowledgments

Many individuals deserve acknowledgment for aiding the successful completion of this textbook. First, I would like to recognize the many graduate students who have sat in my elasticity classes. They are a continual source of challenge and inspiration, and certainly influenced my efforts to find a better way to present this material. A very special recognition goes to one particular student, Ms. Qingli Dai, who developed most of the exercise solutions and did considerable proofreading. Several photoelastic pictures have been graciously provided by our Dynamic Photomechanics Laboratory. Development and production support from several Elsevier staff was greatly appreciated. I would also like to acknowledge the support of my institution, the University of Rhode Island for granting me a sabbatical leave to complete the text. Finally, a special thank you to my wife, Eve, for being patient with my extended periods of manuscript preparation.

This book is dedicated to the late Professor Marvin Stippes of the University of Illinois, who first showed me the elegance and beauty of the subject. His neatness, clarity, and apparent infinite understanding of elasticity will never be forgotten by his students.

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June 2004

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About the Author

Martin H. Sadd is Professor of Mechanical Engineering & Applied Mechanics at the University of Rhode Island. He received his Ph.D. in Mechanics from the Illinois Institute of Technology in 1971 and then began his academic career at Mississippi State University. In 1979 he joined the faculty at Rhode Island and served as department chair from 1991-2000. Dr. Sadd's teaching background is in the area of solid mechanics with emphasis in elasticity, continuum mechanics, wave propagation, and computational methods. He has taught elasticity at two academic institutions, several industries, and at a government laboratory. Professor Sadd's research has been in the area of computational modeling of materials under static and dynamic loading conditions using finite, boundary, and discrete element methods. Much of his work has involved micromechanical modeling of geomaterials including granular soil, rock, and concretes. He has authored over 70 publications and has given numerous presentations at national and international meetings.

Part I Foundations and Elementary Applications

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1 Mathematical Preliminaries

Similar to other field theories such as fluid mechanics, heat conduction, and electromagnetics, the study and application of elasticity theory requires knowledge of several areas of applied mathematics. The theory is formulated in terms of a variety of variables including scalar, vector, and tensor fields, and this calls for the use of tensor notation along with tensor algebra and calculus. Through the use of particular principles from continuum mechanics, the theory is developed as a system of partial differential field equations that are to be solved in a region of space coinciding with the body under study. Solution techniques used on these field equations commonly employ Fourier methods, variational techniques, integral transforms, complex variables, potential theory, finite differences, and finite and boundary elements. Therefore, to develop proper formulation methods and solution techniques for elasticity problems, it is necessary to have an appropriate mathematical background. The purpose of this initial chapter is to provide a background primarily for the formulation part of our study. Additional review of other mathematical topics related to problem solution technique is provided in later chapters where they are to be applied.

1.1 Scalar, Vector, Matrix, and Tensor Definitions

Elasticity theory is formulated in terms of many different types of variables that are either specified or sought at spatial points in the body under study. Some of these variables are *scalar quantities,* representing a single magnitude at each point in space. Common examples include the material density ρ and material moduli such as Young's modulus E, Poisson's ratio ν, or the shear modulus μ. Other variables of interest are *vector quantities* that are expressible in terms of components in a two- or three-dimensional coordinate system. Examples of vector variables are the displacement and rotation of material points in the elastic continuum. Formulations within the theory also require the need for *matrix variables,* which commonly require more than three components to quantify. Examples of such variables include stress and strain. As shown in subsequent chapters, a three-dimensional formulation requires nine components (only six are independent) to quantify the stress or strain at a point. For this case, the variable is normally expressed in a matrix format with three rows and three columns. To summarize this discussion, in a three-dimensional Cartesian coordinate system, scalar, vector, and matrix variables can thus be written as follows:

$$\text{mass density scalar} = \rho$$

$$\text{displacement vector} = \boldsymbol{u} = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + w\boldsymbol{e}_3$$

$$\text{stress matrix} = [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are the usual unit basis vectors in the coordinate directions. Thus, scalars, vectors, and matrices are specified by one, three, and nine components, respectively.

The formulation of elasticity problems not only involves these types of variables, but also incorporates additional quantities that require even more components to characterize. Because of this, most field theories such as elasticity make use of a *tensor formalism* using index notation. This enables efficient representation of all variables and governing equations using a single standardized scheme. The tensor concept is defined more precisely in a later section, but for now we can simply say that scalars, vectors, matrices, and other higher-order variables can all be represented by tensors of various orders. We now proceed to a discussion on the notational rules of order for the tensor formalism. Additional information on tensors and index notation can be found in many texts such as Goodbody (1982) or Chandrasekharaiah and Debnath (1994).

1.2 Index Notation

Index notation is a shorthand scheme whereby a whole set of numbers (elements or components) is represented by a single symbol with subscripts. For example, the three numbers a_1, a_2, a_3 are denoted by the symbol a_i, where index i will normally have the range 1, 2, 3. In a similar fashion, a_{ij} represents the nine numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}. Although these representations can be written in any manner, it is common to use a scheme related to vector and matrix formats such that

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \; a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1.2.1}$$

In the matrix format, a_{1j} represents the first row, while a_{i1} indicates the first column. Other columns and rows are indicated in similar fashion, and thus the first index represents the row, while the second index denotes the column.

In general a symbol $a_{ij\ldots k}$ with N distinct indices represents 3^N distinct numbers. It should be apparent that a_i and a_j represent the same three numbers, and likewise a_{ij} and a_{mn} signify the same matrix. Addition, subtraction, multiplication, and equality of index symbols are defined in the normal fashion. For example, addition and subtraction are given by

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, \; a_{ij} \pm b_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix} \tag{1.2.2}$$

and scalar multiplication is specified as

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \lambda a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} \tag{1.2.3}$$

The multiplication of two symbols with different indices is called *outer multiplication,* and a simple example is given by

$$a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{1.2.4}$$

The previous operations obey usual commutative, associative, and distributive laws, for example:

$$\begin{aligned} &a_i + b_i = b_i + a_i \\ &a_{ij} b_k = b_k a_{ij} \\ &a_i + (b_i + c_i) = (a_i + b_i) + c_i \\ &a_i (b_{jk} c_l) = (a_i b_{jk}) c_l \\ &a_{ij}(b_k + c_k) = a_{ij} b_k + a_{ij} c_k \end{aligned} \tag{1.2.5}$$

Note that the simple relations $a_i = b_i$ and $a_{ij} = b_{ij}$ imply that $a_1 = b_1,\ a_2 = b_2, \ldots$ and $a_{11} = b_{11}, a_{12} = b_{12}, \ldots$ However, relations of the form $a_i = b_j$ or $a_{ij} = b_{kl}$ have ambiguous meaning because the distinct indices on each term are not the same, and these types of expressions are to be avoided in this notational scheme. In general, the distinct subscripts on all individual terms in an equation should match.

It is convenient to adopt the convention that if a subscript appears twice in the same term, then *summation* over that subscript from one to three is implied; for example:

$$\begin{aligned} a_{ii} &= \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} \\ a_{ij} b_j &= \sum_{j=1}^{3} a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + a_{i3} b_3 \end{aligned} \tag{1.2.6}$$

It should be apparent that $a_{ii} = a_{jj} = a_{kk} = \ldots,$ and therefore the *repeated* subscripts or indices are sometimes called *dummy* subscripts. Unspecified indices that are not repeated are called *free* or *distinct* subscripts. The summation convention may be suspended by underlining one of the repeated indices or by writing *no sum*. The use of three or more repeated indices in the same term (e.g., a_{iii} or $a_{iij} b_{ij}$) has ambiguous meaning and is to be avoided. On a given symbol, the process of setting two free indices equal is called *contraction*. For example, a_{ii} is obtained from a_{ij} by contraction on i and j. The operation of outer multiplication of two indexed symbols followed by contraction with respect to one index from each symbol generates an *inner multiplication*; for example, $a_{ij} b_{jk}$ is an inner product obtained from the outer product $a_{ij} b_{mk}$ by contraction on indices j and m.

A symbol $a_{ij\ldots m\ldots n\ldots k}$ is said to be *symmetric* with respect to index pair mn if

$$a_{ij\ldots m\ldots n\ldots k} = a_{ij\ldots n\ldots m\ldots k} \tag{1.2.7}$$

while it is *antisymmetric* or *skewsymmetric* if

$$a_{ij\ldots m\ldots n\ldots k} = -a_{ij\ldots n\ldots m\ldots k} \tag{1.2.8}$$

Note that if $a_{ij\ldots m\ldots n\ldots k}$ is symmetric in mn while $b_{pq\ldots m\ldots n\ldots r}$ is antisymmetric in mn, then the product is zero:

$$a_{ij\ldots m\ldots n\ldots k} b_{pq\ldots m\ldots n\ldots r} = 0 \tag{1.2.9}$$

A useful identity may be written as

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) = a_{(ij)} + a_{[ij]} \tag{1.2.10}$$

The first term $a_{(ij)} = 1/2(a_{ij} + a_{ji})$ is symmetric, while the second term $a_{[ij]} = 1/2(a_{ij} - a_{ji})$ is antisymmetric, and thus an arbitrary symbol a_{ij} can be expressed as the sum of symmetric and antisymmetric pieces. Note that if a_{ij} is symmetric, it has only six independent components. On the other hand, if a_{ij} is antisymmetric, its diagonal terms a_{ii} (no sum on i) must be zero, and it has only three independent components. Note that since $a_{[ij]}$ has only three independent components, it can be related to a quantity with a single index, for example, a_i (see Exercise 1-14).

1.3 Kronecker Delta and Alternating Symbol

A useful special symbol commonly used in index notational schemes is the *Kronecker delta* defined by

$$\delta_{ij} = \begin{cases} 1, \textit{if}\ \ i = j\ (\textit{no sum}) \\ 0, \textit{if}\ \ i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.3.1}$$

Within usual matrix theory, it is observed that this symbol is simply the unit matrix. Note that the Kronecker delta is a symmetric symbol. Particular useful properties of the Kronecker delta include the following:

$$\begin{aligned} &\delta_{ij} = \delta_{ji} \\ &\delta_{ii} = 3, \delta_{\underline{ii}} = 1 \\ &\delta_{ij}a_j = a_i, \delta_{ij}a_i = a_j \\ &\delta_{ij}a_{jk} = a_{ik}, \delta_{jk}a_{ik} = a_{ij} \\ &\delta_{ij}a_{ij} = a_{ii}, \delta_{ij}\delta_{ij} = 3 \end{aligned} \tag{1.3.2}$$

Another useful special symbol is the *alternating* or *permutation symbol* defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ is an even permutation of } 1,2,3 \\ -1, & \text{if } ijk \text{ is an odd permutation of } 1,2,3 \\ \ \ 0, & \text{otherwise} \end{cases} \tag{1.3.3}$$

Consequently, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1, \varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \ldots = 0$. Therefore, of the 27 possible terms for the alternating symbol, 3 are equal to $+1$, three are equal to -1, and all others are 0. The alternating symbol is antisymmetric with respect to any pair of its indices.

This particular symbol is useful in evaluating determinants and vector cross products, and the determinant of an array a_{ij} can be written in two equivalent forms:

$$\det[a_{ij}] = |a_{ij}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3} \tag{1.3.4}$$

where the first index expression represents the row expansion, while the second form is the column expansion. Using the property

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{1.3.5}$$

another form of the determinant of a matrix can be written as

$$\det[a_{ij}] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr}a_{ip}a_{jq}a_{kr} \tag{1.3.6}$$

1.4 Coordinate Transformations

It is convenient and in fact necessary to express elasticity variables and field equations in several different coordinate systems (see Appendix A). This situation requires the development of particular transformation rules for scalar, vector, matrix, and higher-order variables. This concept is fundamentally connected with the basic definitions of tensor variables and their related tensor transformation laws. We restrict our discussion to transformations only between Cartesian coordinate systems, and thus consider the two systems shown in Figure 1-1. The two Cartesian frames (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) differ only by orientation, and the unit basis vectors for each frame are $\{\boldsymbol{e}_i\} = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}'_i\} = \{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$.

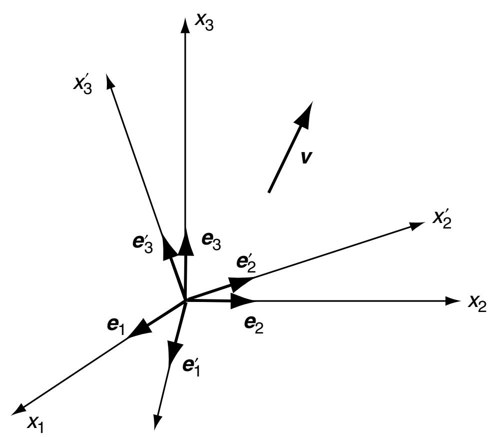

FIGURE 1-1 *Change of Cartesian coordinate frames.*

Let Q_{ij} denote the cosine of the angle between the x_i'-axis and the x_j-axis:

$$Q_{ij} = \cos(x_i', x_j) \tag{1.4.1}$$

Using this definition, the basis vectors in the primed coordinate frame can be easily expressed in terms of those in the unprimed frame by the relations

$$\begin{aligned} \boldsymbol{e}_1' &= Q_{11}\boldsymbol{e}_1 + Q_{12}\boldsymbol{e}_2 + Q_{13}\boldsymbol{e}_3 \\ \boldsymbol{e}_2' &= Q_{21}\boldsymbol{e}_1 + Q_{22}\boldsymbol{e}_2 + Q_{23}\boldsymbol{e}_3 \\ \boldsymbol{e}_3' &= Q_{31}\boldsymbol{e}_1 + Q_{32}\boldsymbol{e}_2 + Q_{33}\boldsymbol{e}_3 \end{aligned} \tag{1.4.2}$$

or in index notation

$$\boldsymbol{e}_i' = Q_{ij}\boldsymbol{e}_j \tag{1.4.3}$$

Likewise, the opposite transformation can be written using the same format as

$$\boldsymbol{e}_i = Q_{ji}\boldsymbol{e}_j' \tag{1.4.4}$$

Now an arbitrary vector $\boldsymbol{v}$ can be written in either of the two coordinate systems as

$$\begin{aligned} \boldsymbol{v} &= v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_i\boldsymbol{e}_i \\ &= v_1'\boldsymbol{e}_1' + v_2'\boldsymbol{e}_2' + v_3'\boldsymbol{e}_3' = v_i'\boldsymbol{e}_i' \end{aligned} \tag{1.4.5}$$

Substituting form (1.4.4) into $(1.4.5)_1$ gives

$$\boldsymbol{v} = v_i Q_{ji}\boldsymbol{e}_j'$$

but from $(1.4.5)_2$, $\boldsymbol{v} = v_j'\boldsymbol{e}_j'$, and so we find that

$$v_i' = Q_{ij}v_j \tag{1.4.6}$$

In similar fashion, using (1.4.3) in $(1.4.5)_2$ gives

$$v_i = Q_{ji}v_j' \tag{1.4.7}$$

Relations (1.4.6) and (1.4.7) constitute the transformation laws for the Cartesian components of a vector under a change of rectangular Cartesian coordinate frame. It should be understood that under such transformations, the vector is unaltered (retaining original length and orientation), and only its components are changed. Consequently, if we know the components of a vector in one frame, relation (1.4.6) and/or relation (1.4.7) can be used to calculate components in any other frame.

The fact that transformations are being made only between orthogonal coordinate systems places some particular restrictions on the transformation or direction cosine matrix Q_{ij}. These can be determined by using (1.4.6) and (1.4.7) together to get

$$v_i = Q_{ji}v_j' = Q_{ji}Q_{jk}v_k \tag{1.4.8}$$

From the properties of the Kronecker delta, this expression can be written as

$$\delta_{ik}v_k = Q_{ji}Q_{jk}v_k \text{ or } (Q_{ji}Q_{jk} - \delta_{ik})v_k = 0$$

and since this relation is true for all vectors v_k, the expression in parentheses must be zero, giving the result

$$Q_{ji}Q_{jk} = \delta_{ik} \tag{1.4.9}$$

In similar fashion, relations (1.4.6) and (1.4.7) can be used to eliminate v_i (instead of v'_i) to get

$$Q_{ij}Q_{kj} = \delta_{ik} \tag{1.4.10}$$

Relations (1.4.9) and (1.4.10) comprise the *orthogonality conditions* that Q_{ij} must satisfy. Taking the determinant of either relation gives another related result:

$$\det[Q_{ij}] = \pm 1 \tag{1.4.11}$$

Matrices that satisfy these relations are called orthogonal, and the transformations given by (1.4.6) and (1.4.7) are therefore referred to as orthogonal transformations.

1.5 Cartesian Tensors

Scalars, vectors, matrices, and higher-order quantities can be represented by a general index notational scheme. Using this approach, all quantities may then be referred to as tensors of different orders. The previously presented transformation properties of a vector can be used to establish the general transformation properties of these tensors. Restricting the transformations to those only between Cartesian coordinate systems, the general set of transformation relations for various orders can be written as

$$\begin{aligned}
&a' = a, \text{ zero order (scalar)} \\
&a'_i = Q_{ip}a_p, \text{ first order (vector)} \\
&a'_{ij} = Q_{ip}Q_{jq}a_{pq}, \text{ second order (matrix)} \\
&a'_{ijk} = Q_{ip}Q_{jq}Q_{kr}a_{pqr}, \text{ third order} \\
&a'_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}a_{pqrs}, \text{ fourth order} \\
&\vdots \\
&a'_{ijk\ldots m} = Q_{ip}Q_{jq}Q_{kr}\cdots Q_{mt}a_{pqr\ldots t} \text{ general order}
\end{aligned} \tag{1.5.1}$$

Note that, according to these definitions, a scalar is a zero-order tensor, a vector is a tensor of order one, and a matrix is a tensor of order two. Relations (1.5.1) then specify the transformation rules for the components of Cartesian tensors of any order under the rotation Q_{ij}. This transformation theory proves to be very valuable in determining the displacement, stress, and strain in different coordinate directions. Some tensors are of a special form in which their components remain the same under all transformations, and these are referred to as *isotropic tensors*. It can be easily verified (see Exercise 1-8) that the Kronecker delta δ_{ij} has such a property and is therefore a second-order isotropic tensor. The alternating symbol ε_{ijk} is found to be the third-order isotropic form. The fourth-order case (Exercise 1-9) can be expressed in terms of products of Kronecker deltas, and this has important applications in formulating isotropic elastic constitutive relations in Section 4.2.

The distinction between the components and the tensor should be understood. Recall that a vector $\boldsymbol{v}$ can be expressed as

$$\begin{aligned} \boldsymbol{v} &= v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_i\boldsymbol{e}_i \\ &= v_1'\boldsymbol{e}_1' + v_2'\boldsymbol{e}_2' + v_3'\boldsymbol{e}_3' = v_i'\boldsymbol{e}_i' \end{aligned} \tag{1.5.2}$$

In a similar fashion, a second-order tensor $\boldsymbol{A}$ can be written

$$\begin{aligned} \boldsymbol{A} = &A_{11}\boldsymbol{e}_1\boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1\boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1\boldsymbol{e}_3 \\ &+ A_{21}\boldsymbol{e}_2\boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\boldsymbol{e}_3 \\ &+ A_{31}\boldsymbol{e}_3\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\boldsymbol{e}_3 \\ &= A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = A'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j \end{aligned} \tag{1.5.3}$$

and similar schemes can be used to represent tensors of higher order. The representation used in equation (1.5.3) is commonly called *dyadic notation,* and some authors write the dyadic products $\boldsymbol{e}_i\boldsymbol{e}_j$ using a *tensor product* notation $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Additional information on dyadic notation can be found in Weatherburn (1948) and Chou and Pagano (1967).

Relations (1.5.2) and (1.5.3) indicate that any tensor can be expressed in terms of components in any coordinate system, and it is only the components that change under coordinate transformation. For example, the state of stress at a point in an elastic solid depends on the problem geometry and applied loadings. As is shown later, these stress components are those of a second-order tensor and therefore obey transformation law $(1.5.1)_3$. Although the components of the stress tensor change with the choice of coordinates, the stress tensor (representing the state of stress) does not.

An important property of a tensor is that if we know its components in one coordinate system, we can find them in any other coordinate frame by using the appropriate transformation law. Because the components of Cartesian tensors are representable by indexed symbols, the operations of equality, addition, subtraction, multiplication, and so forth are defined in a manner consistent with the indicial notation procedures previously discussed. The terminology *tensor* without the adjective *Cartesian* usually refers to a more general scheme in which the coordinates are not necessarily rectangular Cartesian and the transformations between coordinates are not always orthogonal. Such general tensor theory is not discussed or used in this text.

EXAMPLE 1-1: Transformation Examples

The components of a first- and second-order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \; a_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

EXAMPLE 1-1: Transformation Examples–*Cont'd*

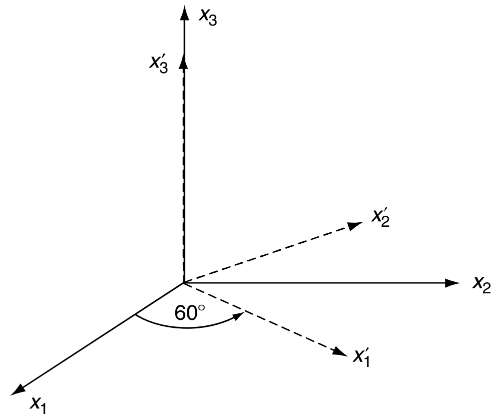

FIGURE 1-2 *Coordinate transformation.*

Determine the components of each tensor in a new coordinate system found through a rotation of 60° ($\pi/6$ radians) about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis (see Figure 1-2).

The original and primed coordinate systems shown in Figure 1-2 establish the angles between the various axes. The solution starts by determining the rotation matrix for this case:

$$Q_{ij} = \begin{bmatrix} \cos 60^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 150^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation for the vector quantity follows from equation $(1.5.1)_2$:

$$a_i' = Q_{ij}a_j = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 + 2\sqrt{3} \\ 2 - \sqrt{3}/2 \\ 2 \end{bmatrix}$$

and the second-order tensor (matrix) transforms according to $(1.5.1)_3$:

$$\begin{aligned} a_{ij}' = Q_{ip}Q_{jq}a_{pq} &= \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2 + \sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1 - 3\sqrt{3}/2 \\ 3/2 + \sqrt{3} & 1 - 3\sqrt{3}/2 & 4 \end{bmatrix} \end{aligned}$$

where $[\ ]^T$ indicates transpose (defined in Section 1.7). Although simple transformations can be worked out by hand, for more general cases it is more convenient to use a computational scheme to evaluate the necessary matrix multiplications required in the transformation laws (1.5.1). MATLAB software is ideally suited to carry out such calculations, and an example program to evaluate the transformation of second-order tensors is given in Example C-1 in Appendix C.

1.6 Principal Values and Directions for Symmetric Second-Order Tensors

Considering the tensor transformation concept previously discussed, it should be apparent that there might exist particular coordinate systems in which the components of a tensor take on maximum or minimum values. This concept is easily visualized when we consider the components of a vector shown in Figure 1-1. If we choose a particular coordinate system that has been rotated so that the x_3-axis lies along the direction of the vector, then the vector will have components $\boldsymbol{v} = \{0, 0, |\boldsymbol{v}|\}$. For this case, two of the components have been reduced to zero, while the remaining component becomes the largest possible (the total magnitude).

This situation is most useful for symmetric second-order tensors that eventually represent the stress and/or strain at a point in an elastic solid. The direction determined by the unit vector $\boldsymbol{n}$ is said to be a *principal direction* or *eigenvector* of the symmetric second-order tensor a_{ij} if there exists a parameter λ such that

$$a_{ij}n_j = \lambda n_i \tag{1.6.1}$$

where λ is called the *principal value* or *eigenvalue* of the tensor. Relation (1.6.1) can be rewritten as

$$(a_{ij} - \lambda\delta_{ij})n_j = 0$$

and this expression is simply a homogeneous system of three linear algebraic equations in the unknowns n_1, n_2, n_3. The system possesses a nontrivial solution if and only if the determinant of its coefficient matrix vanishes, that is:

$$\det[a_{ij} - \lambda\delta_{ij}] = 0$$

Expanding the determinant produces a cubic equation in terms of λ:

$$\det[a_{ij} - \lambda\delta_{ij}] = -\lambda^3 + I_a\lambda^2 - II_a\lambda + III_a = 0 \tag{1.6.2}$$

where

$$\begin{aligned} I_a &= a_{ii} = a_{11} + a_{22} + a_{33} \\ II_a &= \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \\ III_a &= \det[a_{ij}] \end{aligned} \tag{1.6.3}$$

The scalars I_a, II_a, and III_a are called the *fundamental invariants* of the tensor a_{ij}, and relation (1.6.2) is known as the *characteristic equation*. As indicated by their name, the three invariants do not change value under coordinate transformation. The roots of the characteristic equation determine the allowable values for λ, and each of these may be back-substituted into relation (1.6.1) to solve for the associated principal direction $\boldsymbol{n}$.

Under the condition that the components a_{ij} are real, it can be shown that all three roots λ_1, λ_2, λ_3 of the cubic equation (1.6.2) must be real. Furthermore, if these roots are distinct, the principal directions associated with each principal value are orthogonal. Thus, we can conclude that every symmetric second-order tensor has at least three mutually perpendicular principal directions and at most three distinct principal values that are the roots of the characteristic equation. By denoting the principal directions $\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(2)}$, $\boldsymbol{n}^{(3)}$ corresponding to the principal values λ_1, λ_2, λ_3, three possibilities arise:

1. All three principal values distinct; thus, the three corresponding principal directions are unique (except for sense).
2. Two principal values equal ($\lambda_1 \neq \lambda_2 = \lambda_3$); the principal direction $\boldsymbol{n}^{(1)}$ is unique (except for sense), and every direction perpendicular to $\boldsymbol{n}^{(1)}$ is a principal direction associated with λ_2, λ_3.
3. All three principal values equal; every direction is principal, and the tensor is isotropic, as per discussion in the previous section.

Therefore, according to what we have presented, it is always possible to identify a right-handed Cartesian coordinate system such that each axis lies along the principal directions of any given symmetric second-order tensor. Such axes are called the *principal axes* of the tensor. For this case, the basis vectors are actually the unit principal directions $\{\boldsymbol{n}^{(1)}, \boldsymbol{n}^{(2)}, \boldsymbol{n}^{(3)}\}$, and it can be shown that with respect to principal axes the tensor reduces to the diagonal form

$$a_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.6.4}$$

Note that the fundamental invariants defined by relations (1.6.3) can be expressed in terms of the principal values as

$$\begin{aligned} I_a &= \lambda_1 + \lambda_2 + \lambda_3 \\ II_a &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ III_a &= \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{1.6.5}$$

The eigenvalues have important extremal properties. If we arbitrarily rank the principal values such that $\lambda_1 > \lambda_2 > \lambda_3$, then λ_1 will be the largest of all possible diagonal elements, while λ_3 will be the smallest diagonal element possible. This theory is applied in elasticity as we seek the largest stress or strain components in an elastic solid.

EXAMPLE 1-2: Principal Value Problem

Determine the invariants and principal values and directions of the following symmetric second-order tensor:

$$a_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

The invariants follow from relations (1.6.3)

Continued

EXAMPLE 1-2: Principal Value Problem–*Cont'd*

$$I_a = a_{ii} = 2 + 3 - 3 = 2$$

$$II_a = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = 6 - 25 - 6 = -25$$

$$III_a = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{vmatrix} = 2(-9 - 16) = -50$$

The characteristic equation then becomes

$$\det[a_{ij} - \lambda\delta_{ij}] = -\lambda^3 + 2\lambda^2 + 25\lambda - 50 = 0$$
$$\Rightarrow (\lambda - 2)(\lambda^2 - 25) = 0$$
$$\therefore \lambda_1 = 5, \lambda_2 = 2, \lambda_3 = -5$$

Thus, for this case all principal values are distinct.

For the $\lambda_1 = 5$ root, equation (1.6.1) gives the system

$$-3n_1^{(1)} = 0$$
$$-2n_2^{(1)} + 4n_3^{(1)} = 0$$
$$4n_2^{(1)} - 8n_3^{(1)} = 0$$

which gives a normalized solution $\boldsymbol{n}^{(1)} = \pm(2\boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{5}$. In similar fashion, the other two principal directions are found to be $\boldsymbol{n}^{(2)} = \pm\boldsymbol{e}_1$, $\boldsymbol{n}^{(3)} = \pm(\boldsymbol{e}_2 - 2\boldsymbol{e}_3)/\sqrt{5}$. It is easily verified that these directions are mutually orthogonal. Figure 1-3 illustrates their directions with respect to the given coordinate system, and this establishes the right-handed principal coordinate axes (x_1', x_2', x_3'). For this case, the transformation matrix Q_{ij} defined by (1.4.1) becomes

$$Q_{ij} = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

Notice the eigenvectors actually form the rows of the Q-matrix.

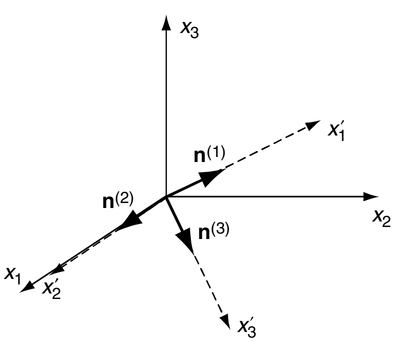

FIGURE 1-3 *Principal axes for Example 1-2.*

EXAMPLE 1-2: Principal Value Problem–*Cont'd*

Using this in the transformation law $(1.5.1)_3$, the components of the given second-order tensor become

$$a'_{ij} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

This result then validates the general theory given by relation (1.6.4) indicating that the tensor should take on diagonal form with the principal values as the elements.

Only simple second-order tensors lead to a characteristic equation that is factorable, thus allowing solution by hand calculation. Most other cases normally develop a general cubic equation and a more complicated system to solve for the principal directions. Again particular routines within the MATLAB package offer convenient tools to solve these more general problems. Example C-2 in Appendix C provides a simple code to determine the principal values and directions for symmetric second-order tensors.

1.7 Vector, Matrix, and Tensor Algebra

Elasticity theory requires the use of many standard algebraic operations among vector, matrix, and tensor variables. These operations include dot and cross products of vectors and numerous matrix/tensor products. All of these operations can be expressed efficiently using compact tensor index notation. First, consider some particular vector products. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, with Cartesian components a_i and b_i, the *scalar* or *dot product* is defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i \tag{1.7.1}$$

Because all indices in this expression are repeated, the quantity must be a scalar, that is, a tensor of order zero. The magnitude of a vector can then be expressed as

$$|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2} = (a_i a_i)^{1/2} \tag{1.7.2}$$

The *vector* or *cross product* between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} a_j b_k \boldsymbol{e}_i \tag{1.7.3}$$

where $\boldsymbol{e}_i$ are the unit basis vectors for the coordinate system. Note that the cross product gives a vector resultant whose components are $\varepsilon_{ijk} a_j b_k$. Another common vector product is the *scalar triple product* defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k \tag{1.7.4}$$

Next consider some common matrix products. Using the usual direct notation for matrices and vectors, common products between a matrix $\boldsymbol{A}= [\boldsymbol{A}]$ with a vector $\boldsymbol{a}$ can be written as