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Lecture Notes in:

MATRIX STRUCTURAL ANALYSIS
with an
Introduction to Finite Elements

CVEN4525/5525

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