April 18, 2007

Open channel hydraulics

John Fenton

Abstract

This course of 15 lectures provides an introduction to open channel hydraulics, the generic name for the study of flows in rivers, canals, and sewers, where the distinguishing characteristic is that the surface is unconfined. This means that the location of the surface is also part of the problem, and allows for the existence of waves – generally making things more interesting!

At the conclusion of this subject students will understand the nature of flows and waves in open channels and be capable of solving a wide range of commonly encountered problems.

Table of Contents

References

Ackers, P., White, W. R., Perkins, J. A. & Harrison, A. J. M. (1978) *Weirs and Flumes for Flow Measurement*, Wiley.

Boiten, W. (2000) *Hydrometry*, Balkema.

Bos, M. G. (1978) *Discharge Measurement Structures*, Second Edn, International Institute for Land Reclamation and Improvement, Wageningen.

Fenton, J. D. (2002) The application of numerical methods and mathematics to hydrography, *in Proc. 11th Australasian Hydrographic Conference, Sydney, 3 July - 6 July 2002.*

Fenton, J. D. & Abbott, J. E. (1977) Initial movement of grains on a stream bed: the effect of relative protrusion, *Proc. Roy. Soc. Lond. A* **352**, 523–537.

French, R. H. (1985) *Open-Channel Hydraulics*, McGraw-Hill, New York.

Henderson, F. M. (1966) *Open Channel Flow*, Macmillan, New York.

Herschy, R. W. (1995) *Streamflow Measurement*, Second Edn, Spon, London.

Jaeger, C. (1956) *Engineering Fluid Mechanics*, Blackie, London.

Montes, S. (1998) *Hydraulics of Open Channel Flow*, ASCE, New York.

Novak, P., Moffat, A. I. B., Nalluri, C. & Narayanan, R. (2001) *Hydraulic Structures*, Third Edn, Spon, London.

Yalin, M. S. & Ferreira da Silva, A. M. (2001) *Fluvial Processes*, IAHR, Delft.

Useful references

The following table shows some of the many references available, which the lecturer may refer to in these notes, or which students might find useful for further reading. For most books in the list, The University of Melbourne Engineering Library Reference Numbers are given.

Reference	**Comments**
Bos, M. G. (1978), *Discharge Measurement Structures*, second edn, International Institute for Land Reclamation and Improvement, Wageningen.	Good encyclopaedic treatment of structures
Bos, M. G., Replogle, J. A. & Clemmens, A. J. (1984), *Flow Measuring Flumes for Open Channel Systems*, Wiley.	Good encyclopaedic treatment of structures
Chanson, H. (1999), *The Hydraulics of Open Channel Flow*, Arnold, London.	Good technical book, moderate level, also sediment aspects
Chaudhry, M. H. (1993), *Open-channel flow*, Prentice-Hall.	Good technical book
Chow, V. T. (1959), *Open-channel Hydraulics*, McGraw-Hill, New York.	Classic, now dated, not so readable
Dooge, J. C. I. (1992) , The Manning formula in context, *in* B. C. Yen, ed., *Channel Flow Resistance: Centennial of Manning's Formula*, Water Resources Publications, Littleton, Colorado, pp. 136–185.	Interesting history of Manning's law
Fenton, J. D. & Keller, R. J. (2001), The calculation of streamflow from measurements of stage, Technical Report 01/6, Co-operative Research Centre for Catchment Hydrology, Monash University.	Two level treatment - practical aspects plus high level review of theory
Francis, J. & Minton, P. (1984), *Civil Engineering Hydraulics*, fifth edn, Arnold, London.	Good elementary introduction
French, R. H. (1985), *Open-Channel Hydraulics*, McGraw-Hill, New York.	Wide general treatment
Henderson, F. M. (1966), *Open Channel Flow*, Macmillan, New York.	Classic, high level, readable
Hicks, D. M. & Mason, P. D. (1991) , *Roughness Characteristics of New Zealand Rivers*, DSIR Marine and Freshwater, Wellington.	Interesting presentation of Manning's n for different streams
Jain, S. C. (2001), *Open-Channel Flow*, Wiley.	High level, but terse and readable
Montes, S. (1998), *Hydraulics of Open Channel Flow*, ASCE, New York.	Encyclopaedic
Novak, P., Moffat, A. I. B., Nalluri, C. & Narayanan, R. (2001), *Hydraulic Structures*, third edn, Spon, London.	Standard readable presentation of structures
Townson, J. M. (1991), *Free-surface Hydraulics*, Unwin Hyman, London.	Simple, readable, mathematical

1. Introduction

The flow of water with an unconfined free surface at atmospheric pressure presents some of the most common problems of fluid mechanics to civil and environmental engineers. Rivers, canals, drainage canals, floods, and sewers provide a number of important applications which have led to the theories and methods of open channel hydraulics. The main distinguishing characteristic of such studies is that the location of the surface is also part of the problem. This allows the existence of waves, both stationary and travelling. In most cases, where the waterway is much longer than it is wide or deep, it is possible to treat the problem as an essentially one-dimensional one, and a number of simple and powerful methods have been developed.

In this course we attempt a slightly more general view than is customary, where we allow for real fluid effects as much as possible by allowing for the variation of velocity over the waterway cross section. We recognise that we can treat this approximately, but it remains an often-unknown aspect of each problem.

This reminds us that we are obtaining approximate solutions to approximate problems, but it does allow some simplifications to be made.

The basic approximation in open channel hydraulics, which is usually a very good one, is that variation along the channel is gradual. One of the most important consequences of this is that the pressure in the water is given by the hydrostatic approximation, that it is proportional to the depth of water above.

In Australia there is a slightly non-standard nomenclature which is often used, namely to use the word ”channel” for a canal, which is a waterway which is usually constructed, and with a uniform section. We will use the more international English convention, that such a waterway is called a canal, and we will use the words ”waterway”, ”stream”, or ”channel” as generic terms which can describe any type of irregular river or regular canal or sewer with a free surface.

1.1 Types of channel flow to be studied

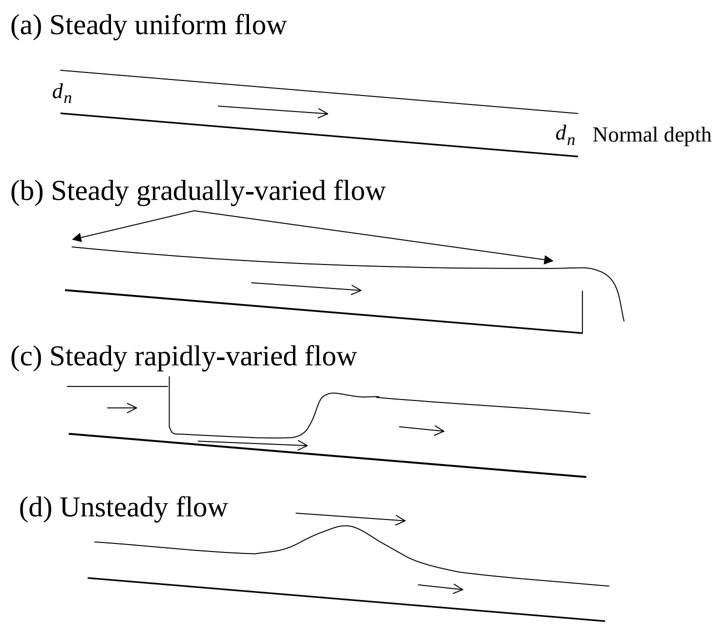

Figure 1-1. Different types of flow in an open channel

Case (a) – Steady uniform flow: Steady flow is where there is no change with time, $\partial/\partial t \equiv 0$. Distant from control structures, gravity and friction are in balance, and if the cross-section is constant, the flow is uniform, $\partial/\partial x \equiv 0$. We will examine empirical laws which predict flow for given bed slope and roughness and channel geometry.

Case (b) – Steady gradually-varied flow: Gravity and friction are in balance here too, but when a control is introduced which imposes a water level at a certain point, the height of the surface varies along the channel for some distance. For this case we will develop the differential equation which describes how conditions vary along the waterway.

Case (c) – Steady rapidly-varied flow: Figure 1-1(c) shows three separate gradually-varied flow states separated by two rapidly-varied regions: (1) flow under a sluice gate and (2) a hydraulic jump. The complete problem as presented in the figure is too difficult for us to study, as the basic hydraulic approximation that variation is gradual and that the pressure distribution is hydrostatic breaks down in the rapid transitions between the different gradually-varied states. We can, however, analyse such problems by considering each of the almost-uniform flow states and consider energy or momentum conservation between them as appropriate. In these sorts of problems we will assume that the slope of the stream balances the friction losses and we treat such problems as frictionless flow over a generally-horizontal bed, so that for the individual states between rapidly-varied regions we usually consider the flow to be uniform and frictionless, so that the whole problem is modelled as a sequence of quasi-uniform flow states.

Case (d) – Unsteady flow: Here conditions vary with time and position as a wave traverses the waterway. We will obtain some results for this problem too.

1.2 Properties of channel flow

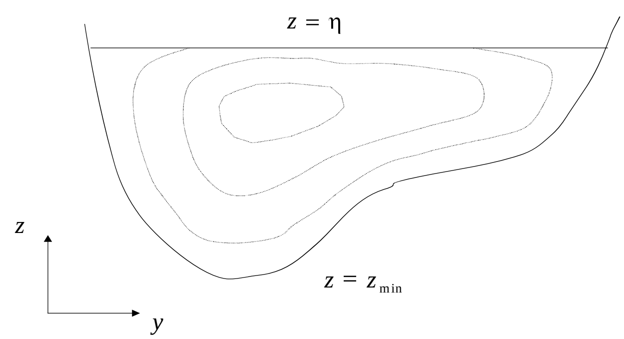

Figure 1-2. Cross-section of flow, showing *isovels*, contours on which velocity normal to the section is constant.

Consider a section of a waterway of arbitrary section, as shown in Figure 1-2. The x co-ordinate is horizontal along the direction of the waterway (normal to the page), y is transverse, and z is vertical. At the section shown the free surface is $z = \eta$, which we have shown to be horizontal across the section, which is a good approximation in many flows.

1.2.1 Discharge across a cross-section

The volume flux or discharge Q at any point is

$$Q = \int_A u \, dA = UA$$

where u is the velocity component in the x or downstream direction, and A is the cross-sectional area. This equation defines the mean horizontal velocity over the section U. In most hydraulic applications the discharge is a more important quantity than the velocity, as it is the volume of water and its rate of propagation, the discharge, which are important.

1.2.2 A generalisation – net discharge across a control surface

Having obtained the expression for volume flux across a plane surface where the velocity vector is normal to the surface, we introduce a generalisation to a control volume of arbitrary shape bounded by a control surface CS. If $\mathbf{u}$ is the velocity vector at any point throughout the control volume and $\hat{\mathbf{n}}$ is a unit vector with direction normal to and directed outwards from a point on the control surface, then $\mathbf{u} \cdot \hat{\mathbf{n}}$ on the control surface is the component of velocity normal to the control surface. If dS is an elemental area of the control surface, then the rate at which fluid volume is leaving across the control surface over that elemental area is $\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$, and integrating gives

$$\text{Total rate at which fluid volume is leaving across the control surface} = \int_{\text{CS}} \mathbf{u}\cdot\hat{\mathbf{n}}\,dS. \quad (1.1)$$

If we consider a finite length of channel as shown in Figure 1-3, with the control surface made up of

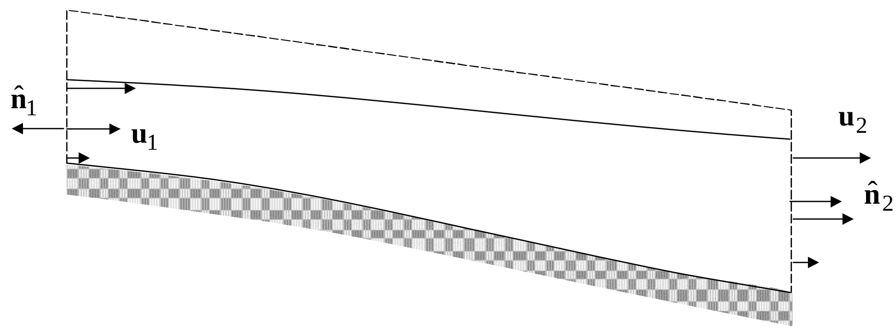

Figure 1-3. Section of waterway and control surface with vertical ends

the bed of the channel, two vertical planes across the channel at stations 1 and 2, and an imaginary enclosing surface somewhere above the water level, then if the channel bed is impermeable, $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ there; $\mathbf{u} = \mathbf{0}$ on the upper surface; on the left (upstream) vertical plane $\mathbf{u}\cdot\hat{\mathbf{n}} = -u_1$, where u_1 is the horizontal component of velocity (which varies across the section); and on the right (downstream) vertical plane $\mathbf{u}\cdot\hat{\mathbf{n}} = +u_2$. Substituting into equation (1.1) we have

$$\begin{aligned}\text{Total rate at which fluid volume is leaving across the control surface} &= -\int_{A_1} u_1\,dA + \int_{A_2} u_2\,dA \\ &= -Q_1 + Q_2.\end{aligned}$$

If the flow is steady and there is no increase of volume inside the control surface, then the total rate of volume leaving is zero and we have $Q_1 = Q_2$.

While that result is obvious, the results for more general situations are not so obvious, and we will generalise this approach to rather more complicated situations – notably where the water surface in the Control Surface *is* changing.

1.2.3 A further generalisation – transport of other quantities across the control surface

We saw that $\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$ is the *volume* flux through an elemental area – if we multiply by fluid density ρ then $\rho\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$ is the rate at which fluid *mass* is leaving across an elemental area of the control surface, with a corresponding integral over the whole surface. Mass flux is actually more fundamental than volume flux, for volume is not necessarily conserved in situations such as compressible flow where the density varies. However in most hydraulic engineering applications we can consider volume to be conserved.

Similarly we can compute the rate at which almost any physical quantity, vector or scalar, is being transported across the control surface. For example, multiplying the mass rate of transfer by the fluid velocity $\mathbf{u}$ gives the rate at which fluid *momentum* is leaving across the control surface, $\rho\mathbf{u}\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dS$.

1.2.4 The energy equation in integral form for steady flow

Bernoulli's theorem states that:

In steady, frictionless, incompressible flow, the energy per unit mass $p/\rho + gz + V^2/2$ is constant

along a streamline,

where V is the fluid speed, $V^2 = u^2 + v^2 + w^2$, in which (u, v, w) are velocity components in a cartesian co-ordinate system (x, y, z) with z vertically upwards, g is gravitational acceleration, p is pressure and ρ is fluid density. In hydraulic engineering it is usually more convenient to divide by g such that we say that the head $p/\rho g + z + V^2/2g$ is constant along a streamline.

In open channel flows (and pipes too, actually, but this seems never to be done) we have to consider the situation where the energy per unit mass varies across the section (the velocity near pipe walls and channel boundaries is smaller than in the middle while pressures and elevations are the same). In this case we cannot apply Bernoulli's theorem across streamlines. Instead, we use an integral form of the energy equation, although almost universally textbooks then neglect variation across the flow and refer to the governing theorem as "Bernoulli". Here we try not to do that.

The energy equation in integral form can be written for a control volume CV bounded by a control surface CS, where there is no heat added or work done on the fluid in the control volume:

$$\underbrace{\frac{\partial}{\partial t} \int\limits_{\mathrm{CV}} \rho\, e\, dV}_{\text{Rate at which energy is increasing inside the CV}} + \underbrace{\int\limits_{\mathrm{CS}} (p + \rho e)\ \mathbf{u}.\hat{\mathbf{n}}\, dS}_{\text{Rate at which energy is leaving the CS}} = 0, \tag{1.2}$$

where t is time, ρ is density, dV is an element of volume, e is the internal energy per unit mass of fluid, which in hydraulics is the sum of potential and kinetic energies

$$e = gz + \frac{1}{2}\left(u^2 + v^2 + w^2\right),$$

where the velocity vector $\mathbf{u} = (u, v, w)$ in a cartesian coordinate system (x, y, z) with x horizontally along the channel and z upwards, $\hat{\mathbf{n}}$ is a unit vector as above, p is pressure, and dS is an elemental area of the control surface.

Here we consider steady flow so that the first term in equation (1.2) is zero. The equation becomes:

$$\int\limits_{\mathrm{CS}} \left(p + \rho g z + \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right)\ \mathbf{u}.\hat{\mathbf{n}}\, dS = 0.$$

We intend to consider problems such as flows in open channels where there is usually no important contribution from lateral flows so that we only need to consider flow entering across one transverse face of the control surface across a pipe or channel and leaving by another. To do this we have the problem of integrating the contribution over a cross-section denoted by A which we also use as the symbol for the cross-sectional area. When we evaluate the integral over such a section we will take u to be the velocity along the channel, perpendicular to the section, and v and w to be perpendicular to that. The contribution over a section of area A is then $\pm E$, where E is the integral over the cross-section:

$$E = \int\limits_{A} \left(p + \rho g z + \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right)\ u\, dA, \tag{1.3}$$

and we take the $\pm$ depending on whether the flow is leaving/entering the control surface, because $\mathbf{u}.\hat{\mathbf{n}} = \pm u$. In the case of no losses, E is constant along the channel. The quantity $\rho Q E$ is the total rate of energy transmission across the section.

Now we consider the individual contributions:

(a) Velocity head term $\frac{\rho}{2}\int_A \left(u^2 + v^2 + w^2\right)\ u\, dA$

If the flow is swirling, then the v and w components will contribute, and if the flow is turbulent there will be extra contributions as well. It seems that the sensible thing to do is to recognise that all velocity components and velocity fluctuations will be of a scale given by the mean flow velocity in the stream at that point,and so we simply write, for the moment ignoring the coefficient $\rho/2$:

$$\int_A \left(u^2+v^2+w^2\right)\, u\, dA = \alpha U^3 A = \alpha \frac{Q^3}{A^2}, \tag{1.4}$$

which defines α as a coefficient which will be somewhat greater than unity, given by

$$\alpha = \frac{\int_A \left(u^2+v^2+w^2\right)\, u\ dA}{U^3 A}. \tag{1.5}$$

Conventional presentations define it as being merely due to the non-uniformity of velocity distribution across the channel:

$$\alpha = \frac{\int_A u^3\, dA}{U^3 A},$$

however we suggest that is more properly written containing the other velocity components (and turbulent contributions as well, ideally). This coefficient is known as a *Coriolis* coefficient, in honour of the French engineer who introduced it.

Most presentations of open channel theory adopt the approximation that there is no variation of velocity over the section, such that it is assumed that $\alpha = 1$, however that is not accurate. Montes (1998, p27) quotes laboratory measurements over a smooth concrete bed giving values of α of 1.035-1.064, while for rougher boundaries such as earth channels larger values are found, such as 1.25 for irrigation canals in southern Chile and 1.35 in the Rhine River. For compound channels very much larger values may be encountered. It would seem desirable to include this parameter in our work, which we will do.

(b) Pressure and potential head terms

These are combined as

$$\int_A (p+\rho g z)\ u\, dA. \tag{1.6}$$

The approximation we now make, common throughout almost all open-channel hydraulics, is the ”hydrostatic approximation”, that pressure at a point of elevation z is given by

$$p \approx \rho g \times \text{ height of water above} = \rho g\left(\eta - z\right), \tag{1.7}$$

where the free surface directly above has elevation η. This is the expression obtained in hydrostatics for a fluid which is not moving. It is an excellent approximation in open channel hydraulics except where the flow is strongly curved, such as where there are short waves on the flow, or near a structure which disturbs the flow. Substituting equation (1.7) into equation (1.6) gives

$$\rho g \int_A \eta\, u\, dA,$$

for the combination of the pressure and potential head terms. If we make the reasonable assumption that η is constant across the channel the contribution becomes

$$\rho g \eta \int_A u\, dA = \rho g \eta Q,$$

from the definition of discharge Q.

(c) Combined terms

Substituting both that expression and equation (1.4) into (1.3) we obtain

$$E = \rho g Q \left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2}\right), \tag{1.8}$$

which, in the absence of losses, would be constant along a channel. This energy flux across entry and exit faces is that which should be calculated, such that it is weighted with respect to the mass flow rate. Most presentations pretend that one can just apply Bernoulli's theorem, which is really only valid along a streamline. However our results in the end are not much different. We can introduce the concept of the *Mean Total Head* H such that

$$H = \frac{\text{Energy flux}}{g \times \text{Mass flux}} = \frac{E}{g \times \rho Q} = \eta + \frac{\alpha}{2g}\frac{Q^2}{A^2}, \tag{1.9}$$

which has units of length and is easily related to elevation in many hydraulic engineering applications, relative to an arbitrary datum. The integral version, equation (1.8), is more fundamental, although in common applications it is simpler to use the mean total head H, which will simply be referred to as the *head* of the flow. Although almost all presentations of open channel hydraulics assume $\alpha = 1$, we will retain the general value, as a better model of the fundamentals of the problem, which is more accurate, but also is a reminder that although we are trying to model reality better, its value is uncertain to a degree, and so are any results we obtain. In this way, it is hoped, we will maintain a sceptical attitude to the application of theory and ensuing results.

(d) Application to a single length of channel – including energy losses

We will represent energy losses by ΔE. For a length of channel where there are no other entry or exit points for fluid, we have

$$E_{\text{out}} = E_{\text{in}} - \Delta E,$$

giving, from equation (1.8):

$$\rho Q_{\text{out}} \left(g\eta + \frac{\alpha}{2}\frac{Q^2}{A^2} \right)_{\text{out}} = \rho Q_{\text{in}} \left(g\eta + \frac{\alpha}{2}\frac{Q^2}{A^2} \right)_{\text{in}} - \Delta E,$$

and as there is no mass entering or leaving, $Q_{\text{out}} = Q_{\text{in}} = Q$, we can divide through by ρQ and by g, as is common in hydraulics:

$$\left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2} \right)_{\text{out}} = \left(\eta + \frac{\alpha}{2g}\frac{Q^2}{A^2} \right)_{\text{in}} - \Delta H,$$

where we have written $\Delta E = \rho g Q \times \Delta H$, where ΔH is the head loss. In spite of our attempts to use energy flux, as Q is constant and could be eliminated, in this head form the terms appear as they are used in conventional applications appealing to Bernoulli's theorem, but with the addition of the α coefficients.

2. Conservation of energy in open channel flow

In this section and the following one we examine the state of flow in a channel section by calculating the energy and momentum flux at that section, while ignoring the fact that the flow at that section might be slowly changing. We are essentially assuming that the flow is locally uniform – *i.e.* it is constant along the channel, $\partial/\partial x \equiv 0$. This enables us to solve some problems, at least to a first, approximate, order. We can make useful deductions about the behaviour of flows in different sections, and the effects of gates, hydraulic jumps, *etc.*. Often this sort of analysis is applied to parts of a rather more complicated flow, such as that shown in Figure 1-1(c) above, where a gate converts a deep slow flow to a faster shallow flow but with the same energy flux, and then *via* an hydraulic jump the flow can increase dramatically in depth, losing energy through turbulence but with the same momentum flux.

2.1 The head/elevation diagram and alternative depths of flow

Consider a steady ($\partial/\partial t \equiv 0$) flow where any disturbances are long, such that the pressure is hydrostatic. We make a departure from other presentations. Conventionally (beginning with Bakhmeteff in 1912) they introduce a co-ordinate origin at the bed of the stream and introduce the concept of "specific energy", which is actually the head relative to that special co-ordinate origin. We believe that the use of that datum somehow suggests that the treatment and the results obtained are special in some way. Also, for irregular cross-sections such as in rivers, the ”bed” or lowest point of the section is poorly defined, and we want to minimise our reliance on such a point. Instead, we will use an arbitrary datum for the head, as it is in keeping with other areas of hydraulics and open channel theory.

Over an arbitrary section such as in Figure 1-2, from equation (1.9), the head relative to the datum can be written

$$H = \eta + \frac{\alpha Q^2}{2g}\frac{1}{A^2(\eta)}, \tag{2.1}$$

where we have emphasised that the cross-sectional area for a given section is a known function of surface elevation, such that we write $A(\eta)$. A typical graph showing the dependence of H upon η is shown in Figure 2-1, which has been drawn for a particular cross-section and a constant value of discharge Q, such that the coefficient $\alpha Q^2/2g$ in equation (2.1) is constant.

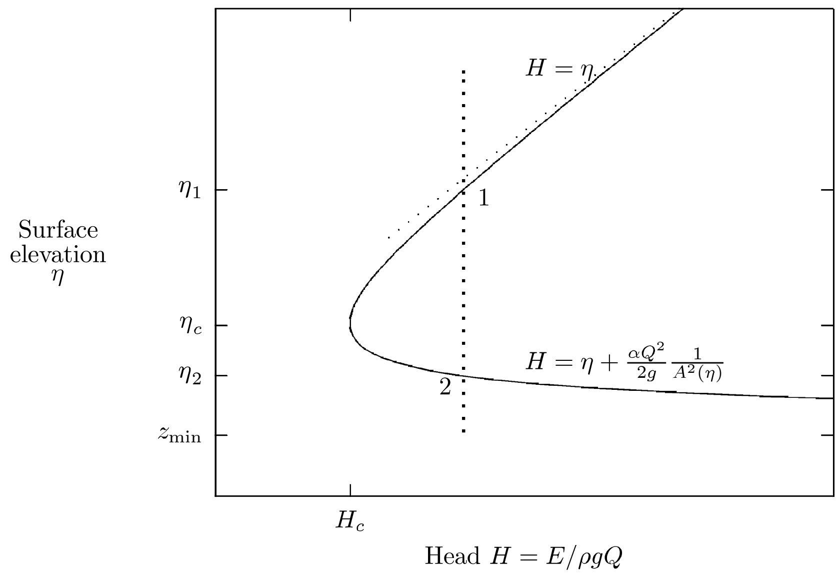

Figure 2-1. Variation of head with surface elevation for a particular cross-section and discharge

The figure has a number of important features, due to the combination of the linear increasing function η and the function $1/A^2(\eta)$ which decreases with η.

- In the shallow flow limit as $\eta \to z_{\min}$ (*i.e.* the depth of flow, and hence the cross-sectional area $A(\eta)$, both go to zero while holding discharge constant) the value of $H \sim \alpha Q^2/2gA^2(\eta)$ becomes very large, and goes to ∞ in the limit.
- In the other limit of deep water, as η becomes large, $H \sim \eta$, as the velocity contribution becomes negligible.
- In between these two limits there is a minimum value of head, at which the flow is called *critical flow*, where the surface elevation is η_c and the head H_c.
- For all other H greater than H_c there are *two* values of depth possible, *i.e.* there are two different flow states possible for the same head.
- The state with the larger depth is called *tranquil*, *slow*, or *sub-critical* flow, where the potential to make waves is relatively small.
- The other state, with smaller depth, of course has faster flow velocity, and is called *shooting*, *fast*, or *super-critical* flow. There is more wave-making potential here, but it is still theoretically possible for the flow to be uniform.
- The two alternative depths for the same discharge and energy have been called *alternate* depths.

That terminology seems to be not quite right – alternate means "occur or cause to occur by turns, go repeatedly from one to another". *Alternative* seems better - "available as another choice", and we will use that.

- In the vicinity of the critical point, where it is easier for flow to pass from one state to another, the flow can very easily form waves (and our hydrostatic approximation would break down).
- Flows can pass from one state to the other. Consider the flow past a sluice gate in a channel as shown in Figure 1-1(c). The relatively deep slow flow passes under the gate, suffering a large reduction in momentum due to the force exerted by the gate and emerging as a shallower faster flow, but with the same energy. These are, for example, the conditions at the points labelled 1 and 2 respectively in Figure 2-1. If we have a flow with head corresponding to that at the point 1 with surface elevation η_1 then the alternative depth is η_2 as shown. It seems that it is not possible to go in the other direction, from super-critical flow to sub-critical flow without some loss of energy, but nevertheless sometimes it is necessary to calculate the corresponding sub-critical depth. The mathematical process of solving either problem, equivalent to reading off the depths on the graph, is one of solving the equation

$$\underbrace{\frac{\alpha Q^2}{2gA^2(\eta_1)} + \eta_1}_{H_1} = \underbrace{\frac{\alpha Q^2}{2gA^2(\eta_2)} + \eta_2}_{H_2} \tag{2.2}$$

for η_2 if η_1 is given, or vice versa. Even for a rectangular section this equation is a nonlinear transcendental equation which has to be solved numerically by procedures such as Newton's method.

2.2 Critical flow

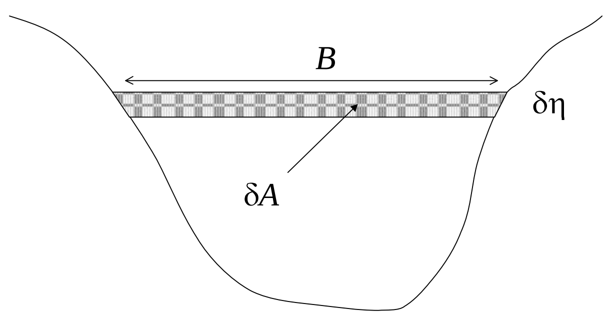

Figure 2-2. Cross-section of waterway with increment of water level

We now need to find what the condition for critical flow is, where the head is a minimum. Equation (2.1) is

$$H = \eta + \frac{\alpha}{2g}\frac{Q^2}{A^2(\eta)},$$

and critical flow is when $dH/d\eta = 0$:

$$\frac{dH}{d\eta} = 1 - \frac{\alpha Q^2}{gA^3(\eta)} \times \frac{dA}{d\eta} = 0.$$

The problem now is to evaluate the derivative $dA/d\eta$. From Figure 2-2, in the limit as $\delta\eta \to 0$ the element of area $\delta A = B\,\delta\eta$,such that $dA/d\eta = B$, the width of the free surface. Substituting, we have *the condition for critical flow*:

$$\alpha\frac{Q^2 B}{gA^3} = 1. \tag{2.3}$$

This can be rewritten as

$$\alpha \frac{(Q/A)^2}{g\,(A/B)} = 1,$$

and as $Q/A = U$, the mean velocity over the section, and $A/B = D$, the *mean* depth of flow, this means that

$$\text{Critical flow occurs when} \quad \alpha \frac{U^2}{gD} = 1, \quad \text{that is, when} \quad \alpha \times \frac{(\text{Mean velocity})^2}{g \times \text{Mean depth}} = 1. \tag{2.4}$$

We write this as

$$\alpha F^2 = 1 \quad \text{or} \quad \sqrt{\alpha} F = 1, \tag{2.5}$$

where the symbol F is the *Froude number*, defined by:

$$F = \frac{Q/A}{\sqrt{gA/B}} = \frac{U}{\sqrt{gD}} = \frac{\text{Mean velocity}}{\sqrt{g \times \text{Mean depth}}}.$$

The usual statement in textbooks is that "critical flow occurs when the Froude number is 1". We have chosen to generalise this slightly by allowing for the coefficient α not necessarily being equal to 1, giving $\alpha F^2 = 1$ at critical flow. Any form of the condition, equation (2.3), (2.4) or (2.5) can be used. The mean depth at which flow is critical is the "critical depth":

$$D_c = \alpha \frac{U^2}{g} = \alpha \frac{Q^2}{gA^2}. \tag{2.6}$$

2.3 The Froude number

The dimensionless Froude number is traditionally used in hydraulic engineering to express the relative importance of inertia and gravity forces, and occurs throughout open channel hydraulics. It is relevant where the water has a free surface. It almost always appears in the form of αF^2 rather than F. It might be helpful here to define F by writing

$$F^2 = \frac{Q^2 B}{gA^3}.$$

Consider a calculation where we attempt to quantify the relative importance of kinetic and potential energies of a flow – and as the depth is the only vertical scale we have we will use that to express the potential energy. We write

$$\frac{\text{Mean kinetic energy per unit mass}}{\text{Mean potential energy per unit mass}} = \frac{\frac{1}{2}\alpha U^2}{gD} = \tfrac{1}{2}\alpha F^2,$$

which indicates something of the nature of the dimensionless number αF^2.

Flows which are fast and shallow have large Froude numbers, and those which are slow and deep have small Froude numbers. For example, consider a river or canal which is $2\,\mathrm{m}$ deep flowing at $0.5\,\mathrm{m\,s^{-1}}$ (make some effort to imagine it - we can well believe that it would be able to flow with little surface disturbance!). We have

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 2}} = 0.11 \quad \text{and} \quad F^2 = 0.012\,,$$

and we can imagine that the rough relative importance of the kinetic energy contribution to the potential contribution really might be of the order of this 1%. Now consider flow in a street gutter after rain. The velocity might also be $0.5\,\mathrm{m\,s^{-1}}$, while the depth might be as little as $2\,\mathrm{cm}$. The Froude number is

$$F = \frac{U}{\sqrt{gD}} \approx \frac{0.5}{\sqrt{10 \times 0.02}} = 1.1 \quad \text{and} \quad F^2 = 1.2\,,$$

which is just super-critical, and we can easily imagine it to have many waves and disturbances on it due to irregularities in the gutter.

It is clear that αF^2 expresses the scale of the importance of kinetic energy to potential energy, even if not in a $1 : 1$ manner (the factor of $1/2$). It seems that αF^2 is a better expression of the relative importance than the traditional use of F. In fact, we suspect that as it always seems to appear in the form $\alpha F^2 = \alpha U^2/gD$, we could define an improved Froude number, $F_{\text{improved}} = \alpha U^2/gD$, which explicitly recognises (a) that U^2/gD is more fundamental than $U/\sqrt{gD}$, and (b) that it is the **weighted** value of u^2 over the whole section, αU^2, which better expresses the importance of dynamic contributions. However, we will use the traditional definition $F = U/\sqrt{gD}$. In tutorials, assignments and exams, unless advised otherwise, you may assume $\alpha = 1$, as has been almost universally done in textbooks and engineering practice. However we will retain α as a parameter in these lecture notes, and we recommend it also in professional practice. Retaining it will, in general, give more accurate results, but also, retaining it while usually not being quite sure of its actual value reminds us that we should not take numerical results as accurately or as seriously as we might. Note that, in the spirit of this, we might well use $g \approx 10$ in practical calculations!

Rectangular channel

There are some special simple features of rectangular channels. These are also applicable to wide channels, where the section properties do not vary much with depth, and they can be modelled by equivalent rectangular channels, or more usually, purely in terms of a unit width. We now find the conditions for critical flow in a rectangular section of breadth b and depth h. We have $A = bh$. From equation (2.3) the condition for critical flow for this section is:

$$\frac{\alpha Q^2}{gb^2h^3} = 1, \tag{2.7}$$

but as $Q = Ubh$, this is the condition

$$\frac{\alpha U^2}{gh} = 1. \tag{2.8}$$

Some useful results follow if we consider the *volume flow per unit width* q:

$$q = \frac{Q}{b} = \frac{Ubh}{b} = Uh. \tag{2.9}$$

Eliminating Q from (2.7) or U from (2.8) or simply using (2.6) with $D_c = h_c$ for the rectangular section gives the critical depth, when H is a minimum:

$$h_c = \left(\alpha\frac{q^2}{g}\right)^{1/3}. \tag{2.10}$$

This shows that the critical depth h_c for rectangular or wide channels depends only on the flow per unit width, and not on any other section properties. As for a rectangular channel it is obvious and convenient to place the origin on the bed, such that $\eta = h$. Then equation (2.1) for critical conditions when H is a minimum, $H = H_c$ becomes

$$H_c = h_c + \frac{\alpha}{2g}\frac{Q^2}{A_c^2} = h_c + \frac{\alpha}{2g}\frac{Q^2}{b^2h_c^2} = h_c + \frac{\alpha q^2}{2g}\frac{1}{h_c^2},$$

and using equation (2.10) to eliminate the q^2 term:

$$H_c = h_c + \frac{h_c^3}{2}\frac{1}{h_c^2} = \frac{3}{2}h_c \quad \text{or,} \quad h_c = \frac{2}{3}H_c. \tag{2.11}$$

2.4 Water level changes at local transitions in channels

Now we consider some simple transitions in open channels from one bed condition to another.

Sub-critical flow over a step in a channel or a narrowing of the channel section: Consider the

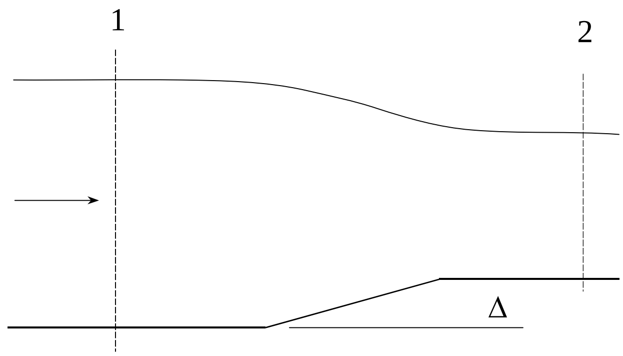

Figure 2-3. Subcritical flow passing over a rise in the bed

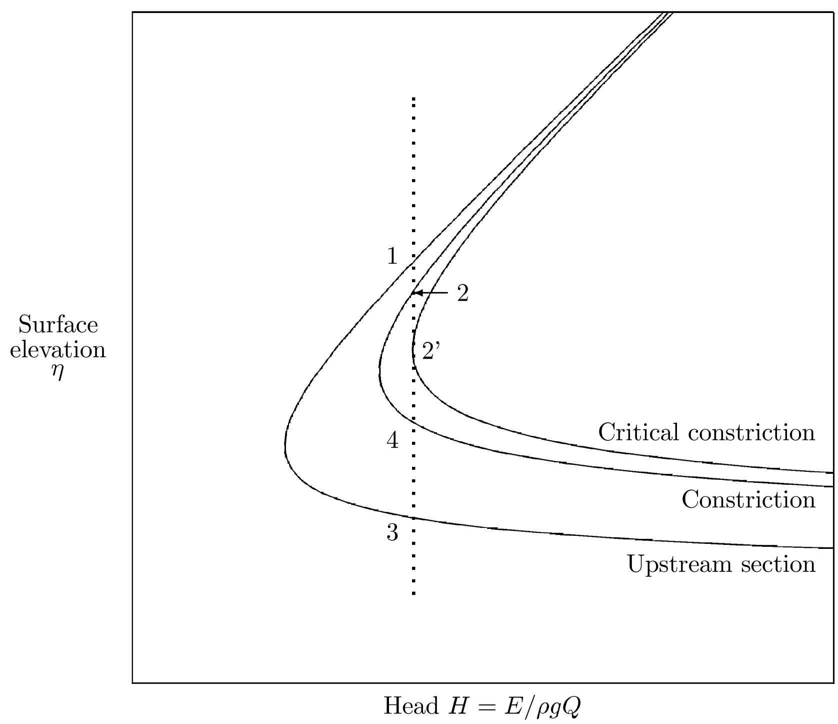

Figure 2-4. Head/Surface-elevation relationships for three cross-sections

flow as shown in Figure 2-3. At the upstream section the (H, η) diagram can be drawn as indicated in Figure 2-4. Now consider another section at an elevation and possible constriction of the channel. The corresponding curve on Figure 2-4 goes to infinity at the higher value of $z_{\min}$ and the curve can be shown to be pushed to the right by this raising of the bed and/or a narrowing of the section. At this stage it is not obvious that the water surface does drop down as shown in Figure 2-3, but it is immediately explained if we consider the point 1 on Figure 2-4 corresponding to the initial conditions. As we assume that no energy is lost in travelling over the channel constriction, the surface level must be as shown at point 2 on Figure 2-4, directly below 1 with the same value of H, and we see how, possibly against expectation, the surface really must drop down if subcritical flow passes through a constriction.

Sub-critical flow over a step or a narrowing of the channel section causing critical flow: Consider now the case where the step Δ is high enough and/or the constriction narrow enough that the previously sub-critical flow is brought to critical, going from point 1 as before, but this time going to point 2' on Figure 2-4. This shows that for the given discharge, the section cannot be constricted more than this amount which would just take it to critical. Otherwise, the (H, η) curve for this section would be moved further to the right and there would be no real depth solutions and no flow possible. In this case the flow in the constriction would remain critical but the upstream depth would have to increase so as to make the flow possible. The step is then acting as a weir, controlling the flow such that there is a unique relationship between flow and depth.

Super-critical flow over a step in a channel or a narrowing of the channel section: Now consider super-critical flow over the same constriction as shown in Figure 2-5. In this case the depth actually increases as the water passes over the step, going from 3 to 4, as the construction in Figure 2-4 shows.

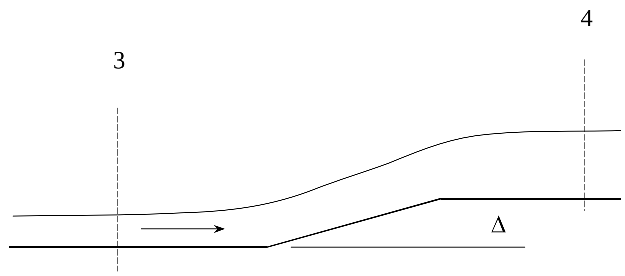

Figure 2-5. Supercritical flow passing over a hump in the bed.

The mathematical problem in each of these cases is to solve an equation similar to (2.2) for η_2, expressing the fact that the head is the same at the two sections:

$$\underbrace{\frac{\alpha Q^2}{2gA_1^2(\eta_1)} + \eta_1}_{H_1} = \underbrace{\frac{\alpha Q^2}{2gA_2^2(\eta_2)} + \eta_2}_{H_2}. \tag{2.12}$$

As the relationship between area and elevation at 2 is different from that at 1, we have shown two different functions for area as a function of elevation, $A_1(\eta_1)$ and $A_2(\eta_2)$.

Example: A rectangular channel of width b_1 carries a flow of Q, with a depth h_1. The channel section is narrowed to a width b_2 and the bed raised by Δ, such that the flow depth above the bed is now h_2. Set up the equation which must be solved for h_2.

Equation (2.12) can be used. If we place the datum on the bed at 1, then $\eta_1 = h_1$ and $A_1(\eta_1) = b_1\eta_1 = b_1 h_1$. Also, $\eta_2 = \Delta + h_2$ and $A_2(\eta_2) = b_2(\eta_2 - \Delta) = b_2 h_2$. The equation becomes

$$\begin{aligned} \frac{\alpha Q^2}{2gb_1^2h_1^2} + h_1 &= \frac{\alpha Q^2}{2gb_2^2h_2^2} + \Delta + h_2, \quad \text{to be solved for } h_2\text{, OR,} \\ \frac{\alpha Q^2}{2gb_1^2h_1^2} + h_1 &= \frac{\alpha Q^2}{2gb_2^2(\eta_2 - \Delta)^2} + \eta_2, \quad \text{to be solved for } \eta_2. \end{aligned}$$

In either case the equation, after multiplying through by h_2 or η_2 respectively, becomes a cubic, which has no simple analytical solution and generally has to be solved numerically. Below we will present methods for this.

2.5 Some practical considerations

2.5.1 Trapezoidal sections